Laplace Distribution Based Stochastic Models

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results based on work of/with Podgorski, Wegener, Aberg, Bolin, Walin

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Outline

1. Introduction
2. Laplace distributions and their generalizations
3. Stochastic processes build upon Laplace distribution
4. Model fitting, estimation
In spatial statistics and geostatistics, **Gaussian random fields** are important tools for constructing models for data. This is mainly because of their simplicity: they are specified through the mean and covariance structures, which is what is needed for **prediction** and **estimation**.
Mathematical Models

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- Drawback: they have **difficulty** in efficiently accounting for **asymmetries** and **unusually extreme values** in the data. When data exhibit such characteristics is usual to consider **transformed Gaussian models**, with the choice of transformation driven by the particular application.
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To provide with a more universal model, there is a growing interest in models featuring more general distributions: *Laplace moving average models*, are formulated as convolutions of Laplace noise and some deterministic kernel and theoretically, they constitute a rich class that is capable of modeling a variety of geometrical asymmetries in the records, and simultaneously can efficiently account for occasional highly extreme events.
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1. Introduction
2. Laplace distributions and their generalizations
3. Stochastic processes build upon Laplace distribution
4. Model fitting, estimation
Asymmetric Laplace distribution

- Characteristic function:

\[ \phi(t) = \frac{e^{it\delta}}{1 + \sigma^2 t^2/2 - \mu t} \]

where \( \delta \) for location, usually \( \delta = 0 \), \( \mu \) shape, \( \sigma > 0 \) the scale parameter.
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- There is an analog of the Central Limit Theorem with random number of terms in the summation: If \(\nu_p\) is a geometric random variable with mean \(1/p\) (waiting until failure) and independent of \(Y_i\)'s which have finite variances, then the random sum

\[
S_p = a_p \sum_{i=1}^{\nu_p} (Y_i + b_p)
\]

converges in distribution to an asymmetric Laplace distribution - stability with respect to geometric summation.
Generalized skewed Laplace distribution

Generalized skewed Laplace distributions have ch.f.

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Left: \( \mu = 0 \)  \quad Right: \( \mu = -\frac{3\sigma}{2^{3/2}} \) and \( \tau = 1/\nu \)
Generalized skewed Laplace distribution, cont.

Equivalent expressions:
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- **Gamma variance-mean mixtures** of normal
  
  \[ \sigma \sqrt{W} Z + \mu W + \delta, \]

  \( Z \) – standard normal, \( W \) – gamma variable with the shape \( \nu \) independent of \( Z \).
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  \( Z \) – standard normal, \( W \) – gamma variable with the shape \( \nu \) independent of \( Z \).

- \[ \kappa W_1 - \frac{1}{\kappa} W_2 + \delta \]
  \( W_1, W_2 \) independent exponential.

The equivalent expressions give rise to different parameterizations! Choice should be based on the purpose, i.e. inference or theoretical properties?
Multivariate extension

- Gamma variance models

\[ Y = \delta + \Gamma \cdot \mu + \Gamma^{1/2} \cdot \Sigma^{1/2} Z \]

where \( \Gamma \) is a gamma variable with the shape parameter \( \nu \) and \( Z \) has a multivariate normal distribution.
Multivariate extension

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  where $\Gamma$ is a gamma variable with the shape parameter $\nu$ and $\mathbf{Z}$ has a multivariate normal distribution.

- Characteristics function
  
  $\phi(\mathbf{t}) = e^{i\delta' \mathbf{t}} \left( \frac{1}{1 + \frac{1}{2} \mathbf{t}' \Sigma \mathbf{t} - i \mathbf{\mu}' \mathbf{t}} \right)^\nu, \quad \mathbf{t} \in \mathbb{R}^d.$
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- Density

\[
\frac{2 \exp(\mu' \Sigma^{-1}(y - \delta))}{(2\pi)^{d/2} \Gamma(\nu) |\Sigma|^{1/2}} \left( \frac{Q(y - \delta)}{C(\Sigma, \mu)} \right)^{\nu - d/2} K_{\nu - d/2}(Q(y - \delta)C(\Sigma, \mu)),
\]

where \( Q(x) = \sqrt{x' \Sigma^{-1} x} \) and \( C(\Sigma, \mu) = \sqrt{2 + \mu' \Sigma^{-1} \mu} \).
Examples of the densities
Stochastic Laplace measure

A stochastic Laplace measure, \( \Lambda \) with parameters \( \delta \in \mathbb{R}, \mu \in \mathbb{R} \) and \( \sigma > 0 \) and controlled by a measure \( m \), is a \( \sigma \)-additive function:

\[
\Lambda : (X, \mathcal{B}, m) \rightarrow L_2(\Omega, \mathcal{F}, \mathbb{P}) :
\]

- \( \Lambda(A) \sim GL(\mu, \sigma, \frac{1}{m(A)}), \quad \forall A \in \mathcal{B} \)
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for disjoint $A_i \in \mathcal{B}$, $\Lambda(A_i)$ are independent and with probability one

$$\Lambda(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \Lambda(A_i).$$
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- Note: $E(\Lambda(A)) = \mu m(A)$, $Var(\Lambda(A)) = (\mu^2 + \sigma^2)m(A)$
Laplace motion

A Laplace motion $L(t)$ with the asymmetry parameter $\mu$, the space scale parameter $\sigma$ and the time scale parameter $\nu$, is defined by the following:

- it starts from the origin, $L(0) = 0$
- it has independent and stationary increments
- $L(t + \nu) - L(t) \sim GL(0, \sigma, 1)$, i.e. has an asymmetric Laplace distribution, centered in mean at zero, with parameter $\mu$ and $\sigma$.

Relation to measure is through

$$\Lambda(a, b] = L(b) - L(a)$$
Alternatively, Laplace motion can be defined through...

- $B(\lambda)$ denote standard Brownian motion

\[ L(\lambda) = B(\Gamma(\lambda)) \]
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- $\Gamma(\lambda)$ a standard gamma process, i.e. Lévy motion with density
  
  \[ g(x) = \frac{1}{\Gamma(\gamma)} x^{\gamma-1} e^{-x}, \quad x > 0 \]

  \[ \gamma = \nu d\lambda \]
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Some properties are inherited from the distribution (stochastic self-similarity):

$\sqrt{\rho}L(\lambda) = L(NB_\rho(\lambda))$,

where $NB_\rho$ is binomial Lévy motion.
Moving averages through stochastic integration

Standard construction of stochastic integrals with deterministic kernels:

\[ X(\tau) = \int_{\mathbb{R}} f(\tau - x) d\Lambda(x). \]
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- Characteristic function
  \[ \phi_X(t) = e^{it\delta} \int f dm \exp \left( - \int_{\mathbb{R}} \log(1 - i\mu tf(x) + \frac{\sigma^2 t^2 f^2(x)}{2}) dm(x) \right). \]

  Extension to multivariate case is immediate.
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- Covariance:
  \[ r(\tau) = \frac{\sigma^2 + \mu^2}{\nu} \int f(x - \tau)f(x) dx \]

- Spectral density:
  \[ R(\omega) = \frac{\sigma^2 + \mu^2}{\nu} \mathcal{F}f(\omega)\mathcal{F}f(-\omega) \]
Simulation

To simulate the Moving Average process ($d = 1$):

$$\Lambda(t) = d \cdot t + \mu \Gamma(t) + B(\Gamma(t)), \quad \Gamma(t) \sim \Gamma\left(\frac{t}{\nu}, 1\right), \ B(t) \sim BM(\sigma)$$
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Very fast and efficient but it looses some resolution where the jumps of the Gamma process occur because of the grid!

For \(d > 1\) we simulate GL increments
Symmetric spatial models - realizations
Asymmetric spatial models
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Fitting the model

The problem of fitting the model is two folded:
Fitting the model

- The problem of fitting the model is two folded:
  - Fit the kernel
Fitting the model

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  - Fit the kernel
  - Fit the Laplace noise
Symmetric kernel case
Symmetric kernel case

A non-parametric approach is to estimate $\hat{f}$ by

$$\hat{f}(x) = \mathcal{F}^{-1} \sqrt{\hat{S}(\omega)},$$

where $\hat{S}(\omega)$ is an estimate of spectrum.
Non-symmetric kernels – measures of asymmetries

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- Various parametrized families of kernels are available.
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- Various parametrized families of kernels are available
- Measures of asymmetries for random surfaces should involve moments of joint distributions of the process and its derivative

Estimation of distributional and kernel parameters has to be performed through solution of the resulting equations.
Laplace moving average trajectories
Measuring tilting using Rice’s formula

- $N(T, A)$ – “number” of times the field $X$ takes value zero in $[0, T]$ and at the same time has a property $A$
Measuring tilting using Rice’s formula

- \( N(T, A) \) – “number” of times the field \( X \) takes value zero in \([0, T]\) and at the same time has a property \( A \)

- For ergodic stationary processes

\[
\lim_{T \to \infty} \frac{N(T, A)}{N(T)} = \frac{\mathbb{E} \left[ \{ X \in A \} \left| \dot{X}(0) \right| X(0) = 0 \right]}{\mathbb{E} \left[ \left| \dot{X}(0) \right| X(0) = 0 \right]},
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\]

- The right hand side represents the biased sampling distribution when sampling is made over the 0-level contour $C_0 = \{\tau : X(\tau) = 0\}$
Tilting of trajectories
Tilting o LMA with odd kernels

$$f(x; \alpha, \beta) = C \cdot \text{sgn}(x) \cdot |x|^{\alpha - 1} e^{-|x|/\beta}, \alpha > 1/2, \beta > 0.$$ Marginal distribution of LMA always symmetric, but trajectories may exhibit some asymmetric features due to asymmetry of kernel.
Measures of tilting

Measures of tilting should involve derivatives of the LMA process. We propose:

\[ \rho_3 = \frac{\mathbb{E} \dot{X}^3}{\mathbb{E}^{3/2} \dot{X}^2}, \]

- skewness of the derivative process. If process has symmetric distribution, this measure fails to capture any asymmetries in the records!
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\[ \rho_{3,1} = \frac{\mathbb{E}(\dot{X}^3 X)}{\mathbb{E}^{3/2} \dot{X}^2 \mathbb{E}^{1/2} X^2} \quad \text{or} \quad r_{3,1} = \frac{\mathbb{E}(\dot{X}^3 X)}{\mathbb{E}^2 X^2} \]
The Gamma kernel

\[ f(x; \tau, \beta) = \frac{2^{\tau-1/2}}{\sqrt{\Gamma(2\tau - 1)}\beta^{\tau-1/2}} x^{\tau-1} e^{-x/\beta} \quad x \geq 0, \tau > 0, \beta > 0, \]

- for \( \tau = 1 \) exponential for \( \tau \to \infty \) Gaussian case.
The Gamma kernel

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- for \( \tau = 1 \) exponential for \( \tau \to \infty \) Gaussian case.

\[ r_{31} = 3\nu \frac{4^{2(2-\tau)}(\tau - 1)\Gamma(4\tau - 6)}{\beta^4\Gamma^2(2\tau - 1)} , \quad \tau > 3/2 \]
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<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( E(\hat{\tau}) )</th>
<th>( Std(\hat{\tau}) )</th>
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<td>.0549</td>
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<tr>
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<tr>
<td>10</td>
<td>11.628</td>
<td>4.364</td>
</tr>
</tbody>
</table>

Monte-Carlo estimation of \( \tau \), for \( \nu = 1 \) and \( \mu = 0 \). Hundred Monte-Carlo replicates of sample size of 300000, with step 0.1.
The Gamma kernel, cont.

\[ \beta = 1 \] and tilting measures \( r_{31} \) (top), and \( \rho_{31} \) (bottom) as functions of \( \tau \).
Distributional parameters - Method of moments

Moments of any integer order are available for LMA - the first four central moments are explicitly available and can be used to fit the four distributional parameters:

\[ E(X) = \mu \cdot \int f dm, \]
\[ E((X - E(X))^2) = \mu^2 + \sigma^2 \cdot \int f^2 dm, \]
\[ E((X - E(X))^3) = \mu (2\mu^2 + 3\sigma^2) \cdot \int f^3 dm, \]
\[ E((X - E(X))^4) = 3(\mu^2 + \sigma^2)^2 \cdot (\int f^2 dm)^2 + 3(2\mu^4 + 4\mu^2 \sigma^2 + \sigma^4) \cdot \int f^4 dm. \]

The method of moments estimation is not the most efficient way of estimation. Empirical moments may fall outside the range permitted by the parameters of the distribution.
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\[
\begin{align*}
\mathbb{E} X &= \mu \cdot \int f dm, \\
\mathbb{E} (X - \mathbb{E} X)^2 &= \left( \mu^2 + \sigma^2 \right) \cdot \int f^2 dm, \\
\mathbb{E} (X - \mathbb{E} X)^3 &= \mu \left( 2\mu^2 + 3\sigma^2 \right) \cdot \int f^3 dm, \\
\mathbb{E} (X - \mathbb{E} X)^4 &= 3 \left( \mu^2 + \sigma^2 \right)^2 \left( \int f^2 dm \right)^2 + 3 \left( 2\mu^4 + 4\mu^2 \sigma^2 + \sigma^4 \right) \cdot \int f^4 dm
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\[ E((X - E(X))^4) = 3 \left( \mu^2 + \sigma^2 \right)^2 \cdot \left( \int f^2 dm \right)^2 + 3 \left( 2\mu^4 + 4\mu^2\sigma^2 + \sigma^4 \right) \cdot \int f^4 dm \]

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Empirical moments may fall outside the range permitted by the parameters of the distribution.
MLE estimation

- System of moment equations that would account for both distributional and kernel parameters maybe fairly complex
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- More efficiently, one can attempt Maximum Likelihood Estimation
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- More efficiently, one can attempt Maximum Likelihood Estimation

- Explicit formula for the MLE estimation can not be obtained except for some very special cases since
  - the density of a generalized Laplace distribution is given through the Bessel function
  - for the Laplace moving averages only characteristics functions can be written in an ‘explicit’ fashion

- Numerical optimization has to be implemented
MLE estimation through EM algorithm

- Representation of GL distribution as $\sigma \sqrt{\Gamma} Z + \mu \Gamma + \delta$ is used and gamma variable $\Gamma$ with parameter $\tau$ is treated as a missing value
MLE estimation through EM algorithm

- Representation of GL distribution as $\sigma \sqrt{\Gamma} Z + \mu \Gamma + \delta$ is used and gamma variable $\Gamma$ with parameter $\tau$ is treated as a missing value.
- For implementation of the algorithm see Podgórski and Wallin, *EM algorithm for maximizing likelihood of linear models involving generalized Laplace distribution*, under preparation.
MME vs. MLE - illustration

The MME estimates of parameters (crosses) vs. the MLE ones (dots). The true parameters $\delta = -0.33$, $\sigma = 1.5$, $\mu = 1.32$, and $\tau = 0.25$ are marked by the large star.
Highlights

Non-Gaussian stochastic fields are proposed that can be suitable for modeling environmental data.

The models are introduced by means of integrals with respect to independently scattered stochastic measures that have generalized Laplace distributions.

Resulting stationary second order processes have, as opposed to their Gaussian counterpart, a possibility of accounting for asymmetry and heavier tails.

Despite this greater flexibility the discussed models still share a lot of spectral properties with Gaussian processes having the latter as a special case.

The models extend directly to random fields.

Spatio-temporal characteristics including asymmetries in the records can be studied by the means of generalized Rice's formula.

Model fitting can be obtained by utilizing: a) Method of Moments (simpler) or b) Maximum Likelihood (more accurate)

Markov Random Fields can be used to localize efficiently the fitting problem and provide a way to account for non-stationarity in the data.
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Everybody believes in the exponential law of errors: the experimenters, because they think it can be proved by mathematics; and the mathematicians, because they believe it has been established by observation."
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