Simulation of non-linear sea waves with Laplace Moving Average processes

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IFREMER
Comportement des Structures en Mer

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Introduction

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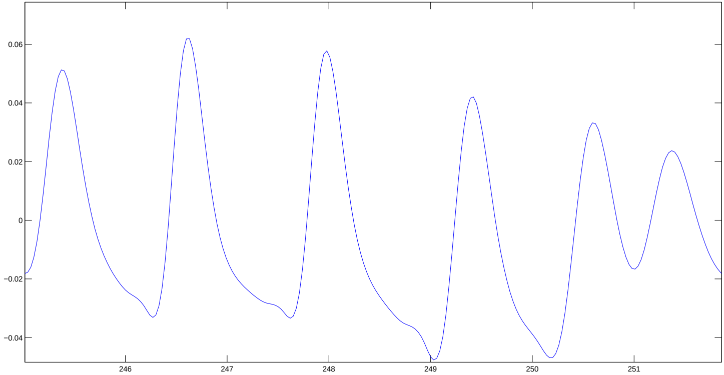

FIGURE : Example of simulated sea waves.

Introduction

- Coastal systems are subject to loadings due to sea waves ;
- In shallow water and with variable bathymetry, thoses waves are highly non-linear with large asymmetries ;
- Numerical methods exist to simulate such data, but are very CPU-consuming, and no long series can be obtained ;
- We will make use of the $\mathcal{LMA}$ introduced recently to model such phenomenons.

Plan

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General Asymmetric Laplace Distribution ($\mathcal{GAL}$)

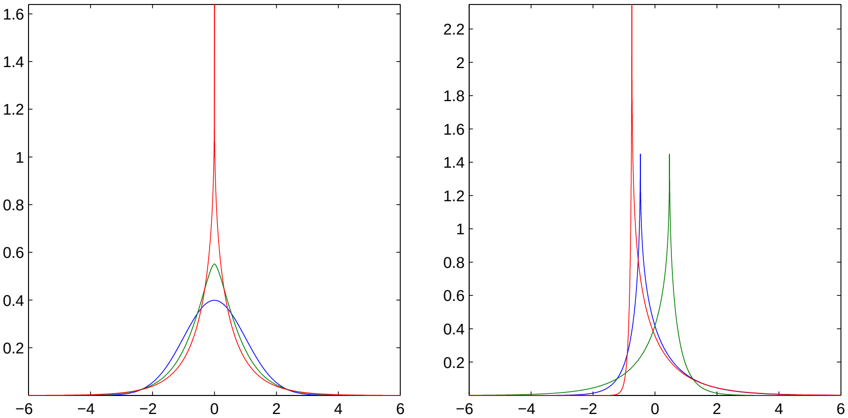
The $\mathcal{GAL}$ distribution is defined thanks to its characteristic function :

$$\Phi(u) = e^{i\delta u}\left(1 - i\mu u + \frac{\sigma^2 u^2}{2}\right)^{-1/\nu}$$

où $\nu,\ \sigma > 0$ et $\delta,\ \mu \in \mathbb{R}$.

- 4 parameters : δ position, μ symmetry, σ scale et ν shape ;
- Entails the Gaussian ($\nu \to 0$) and Gamma ($\nu \to \infty$) cases ;
- Every moments are finite and moreover, for $Y \sim \mathcal{GAL}(\delta, \mu, \sigma, \nu)$:

$$\mathbb{E}Y = \frac{\mu}{\nu} + \delta \qquad \mathbb{V}Y = \frac{\mu^2+\sigma^2}{\nu}$$
$$s = \mu\sqrt{\nu}\frac{2\mu^2+3\sigma^2}{(\mu^2+\sigma^2)^{3/2}} \qquad k_e = 3\nu\left(2 - \frac{\sigma^4}{(\mu^2+\sigma^2)^2}\right)$$



Key facts about the distribution : asymmetric, heavy tailed, .

One can define a Laplace Motion and enter it into a linear filter :

Laplace Moving Average process ($\mathcal{LMA}$)

Let $X_t,\ t \in \mathbb{R}$ be a process such that :

$$X_t = \int_{\mathbb{R}} f(t-x)d\Gamma(x)$$

where Γ is a Laplace noise and f a function (named *kernel*) such that $\int_{\mathbb{R}} f < \infty$ et $\int_{\mathbb{R}} f^2 < \infty$.

Properties

1. $\phi_{X_t}(u) = e^{-iu\mu/\nu \int f} \exp\left\{-\frac{1}{\nu}\int_{\mathbb{R}} \log\left(1 - iu\mu f(x) + \frac{\sigma^2 u^2 f^2(x)}{2}\right) dx\right\};$
2. $r(\tau) = \frac{\sigma^2+\mu^2}{\nu}\int_{\mathbb{R}} f(x-\tau)f(x)dx = \frac{\sigma^2+\mu^2}{\nu}(f * \tilde{f})(\tau)$
3. $S(\omega) = \frac{\sigma^2+\mu^2}{\nu}\left|\mathcal{F}(f)(\omega)\right|^2$

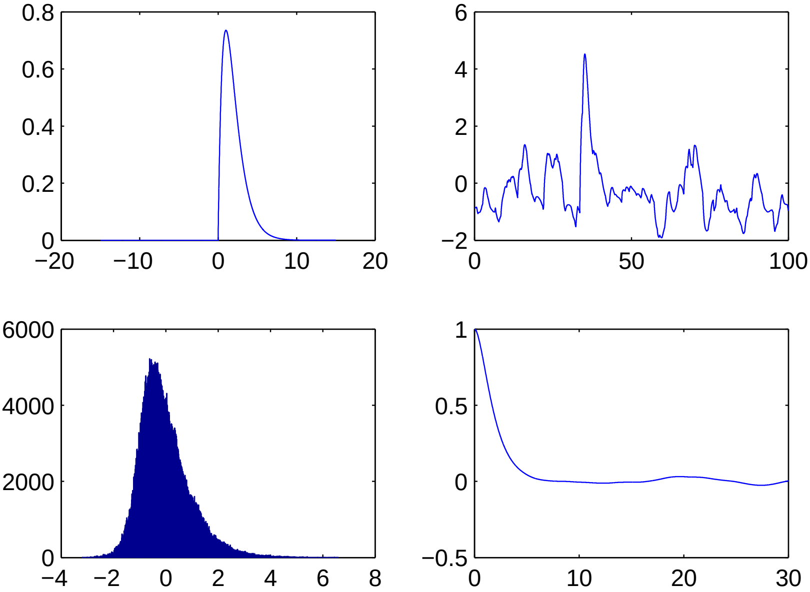
Moments are available :

$$\mathbb{E}X_t = \frac{\mu}{\nu}\int_{\mathbb{R}} f \tag{1}$$

$$\mathbb{V}X_t = \frac{\sigma^2+\mu^2}{\nu}\int_{\mathbb{R}} f^2 \tag{2}$$

$$s = \frac{\mu}{\sqrt{\nu}}\frac{2\mu^2+3\sigma^2}{(\mu^2+\sigma^2)^{3/2}}\frac{\int_{\mathbb{R}} f^3}{(\int_{\mathbb{R}} f^2)^{3/2}} \tag{3}$$

$$\kappa = 3\nu(2-\frac{\sigma^4}{(\mu^2+\sigma^2)^2})\frac{\int_R f^4}{(\int_{\mathbb{R}} f^2)^2} \tag{4}$$


0.8
0.6
0.4
0.2
0
−20
−10
0
10
20
6
4
2
0
−2
0
50
100
6000
4000
2000
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−4
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4
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8
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0.5
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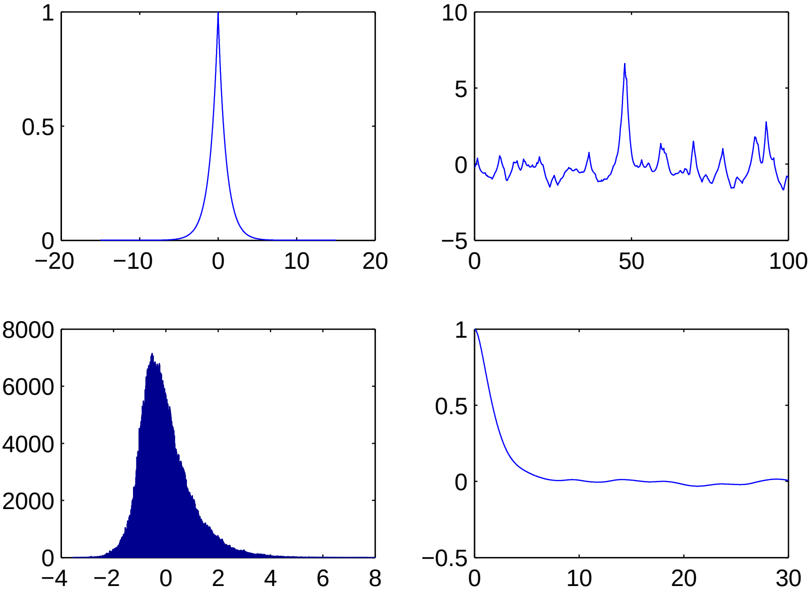
−0.5
0
10
20
30


1
0.5
0
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20
10
5
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−5
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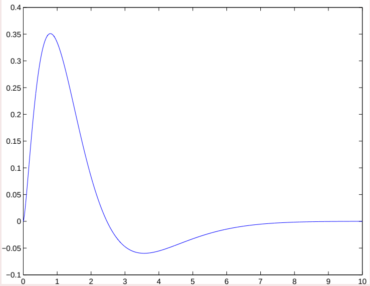
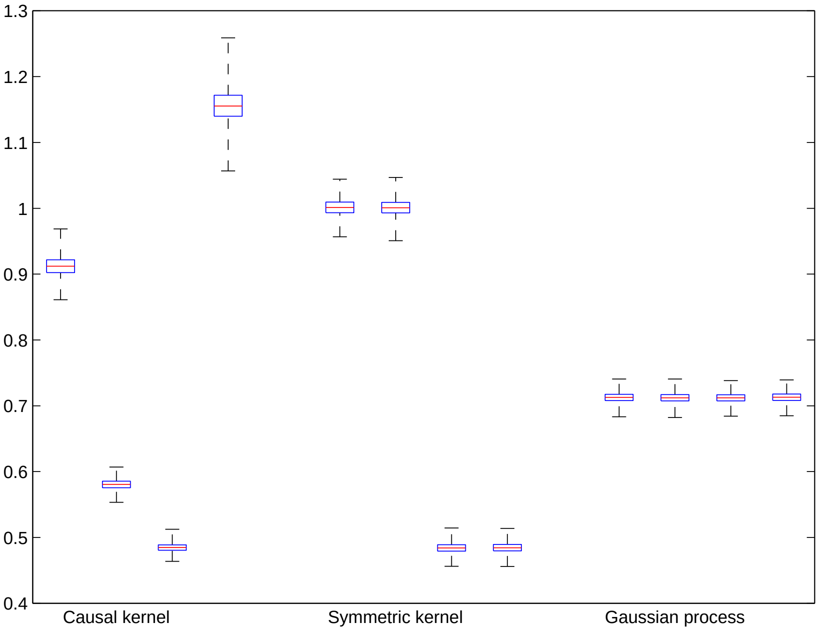
100
8000
6000
4000
2000
0
−4
−2
0
2
4
6
8
1
0.5
0
−0.5
0
10
20
30

Criterion of asymmetry



- Slope before a crest ;
- Slope after a crest ;
- Slope down to a trough ;
- Slope after a trough.


1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
Causal kernel
Symmetric kernel
Gaussian process

High-order spectrum

The second order structure is insufficient to retrieve all the properties of the signal, and thus some more characteristics are needed.

Third order structure

The third order cumulant is defined by :

$$C_3(\tau_1, \tau_2) = \mathbb{E}\left[X_t X_{t+\tau_1} X_{t+\tau_2}\right]$$

and the bispectrum, or third order spectrum is its Fourier transform :

$$B_3(\omega_1, \omega_2) = \mathcal{F}\left[C_3(\tau_1, \tau_2)\right]$$

High-order spectrum for a $\mathcal{LMA}$

Bispectrum of a $\mathcal{LMA}$

For a $\mathcal{LMA}$, it can be shown that :

$$B_3(\omega_1, \omega_2) = sF(\omega_1)F(\omega_2)\overline{F(\omega_1 + \omega_2)},$$

where s is the skewness of the Laplace noise and $F = \mathcal{F}(f)$.

Suppose that :

- $F(\omega) = \mathcal{F}(f) = |F(\omega)|e^{\Phi(\omega)}$;
- $B_3(\omega_1, \omega_2) = |B(\omega_1, \omega_2)|e^{\Psi(\omega_1, \omega_2)}$

Then

- Modulus relations : $|B(\omega_1, \omega_2)| = |F(\omega_1)||F(\omega_2)||F(\omega_1 + \omega_2)|$
- For the Phases : $\Psi(\omega_1, \omega_2) = \Phi(\omega_1) + \Phi(\omega_2) - \Phi(\omega_1 + \omega_2)$

Plan

1. Description des $\mathcal{LMA}$
2. Model fitting
   - Estimation of the marginal parameters
   - Fitting the kernel
3. Application to simulated sea waves
4. Conclusion - Perspectives

Estimation of the marginal parameters

- We first assume that the kernel f is known ;
- The the equations (1)-(4) can be solved be replacing the moments by their empirical counterparts ;
- It can happens there is no solution, and approximate solution must be taken.

Moment condition

Let X be a $\mathcal{LMA}$ with kernel f. Then following relation is verified :

$$\frac{s^2}{k_e} \leq \frac{2(\int f^3)^2}{3\int f^4 . \int f^2} \tag{5}$$

where s is the skewness and k_e the kurtosis excess.

Consequences :

- The whole space (s,k_e) cannot be described by a $\mathcal{LMA}$ and in particular we have $k_e \geq 0$;
- If $\int f^3 = 0$, the distribution of X_t is symmetrical.

Fitting the kernel

The following property holds true : $S(\omega) = \frac{\sigma^2+\mu^2}{\nu} \left|\mathcal{F}(f)(\omega)\right|^2$ where S is the second order spectrum and $\mathcal{F}$ is the Fourier transformation.

Any f like this will satisfy this relation :

$$f(x) = \mathcal{F}^{-1}\left\{\sqrt{S(\omega)}e^{-i\Phi(\omega)}\right\}.$$

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- **Antisymmetric** $f_{\mathrm{AntiSym}}(x) \propto \mathcal{F}^{-1}\left\{\sqrt{S(\omega)}e^{-i\frac{\pi}{2}\operatorname{sgn}\omega}\right\}$, non-causal kernel, antisymmetric ;

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Minimum Phase $f_{\text{MP}}(x) \propto \mathcal{F}^{-1}\left\{\sqrt{S(\omega)}e^{-i\mathcal{H}[\log\sqrt{S(\omega)}]}\right\}$, where $\mathcal{H}$ is the Hilbert transform. $f_{\text{MP}}(x) = 0$ for $x < 0$.

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One needs to find the appropriate Φ.

High-order spectrum for a $\mathcal{LMA}$

Recall some properties :

- $F(\omega) = \mathcal{F}(f) = |F(\omega)|e^{\Phi(\omega)}$;
- $B_3(\omega_1, \omega_2) = |B(\omega_1, \omega_2)|e^{\Psi(\omega_1, \omega_2)}$

Then a relation exists for the phases :

$$\Psi(\omega_1, \omega_2) = \Phi(\omega_1) + \Phi(\omega_2) - \Phi(\omega_1 + \omega_2)$$

This relation can be used to retrieve the unknown Φ from the data, up to a scale and a linear shift.

Parameter	True Kernel	Sym. kern.	Proposed kern.
$\delta = -0.5$	-0.52 (0.08)	-0.79 (0.14)	-0.52 (0.08)
$\mu = 1$	0.96 (0.04)	0.95 (0.07)	0.96 (0.04)
$\sigma = 1$	1.01 (0.18)	0.96 (0.32)	1.01 (0.18)
$\nu = 2$	1.98 (0.30)	1.24 (0.20)	1.98 (0.31)
$\rho_{3,0} = 2.55$	2.46 (0.38)	0.00 (0.00)	2.44 (0.38)
$\rho_{3,1} = 3.27$	3.24 (0.32)	0.31 (0.03)	3.25 (0.33)

TABLE : Estimations of the parameters of a $\mathcal{LMA}$ with minimum phase kernel ($f(x) = 2xe^x \mathbb{1}_{x \geq 0}$). Standard deviation in bracket.

where

$$\rho_{3,0} = \frac{\mathbb{E}\dot{X}^3}{\mathbb{E}^{3/2}\dot{X}^2} \qquad \rho_{3,1} = \frac{\mathbb{E}\dot{X}^3 X}{\mathbb{E}^{3/2}\dot{X}^2 \mathbb{E}^{1/2} X^2}$$

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- Irregular non-linear sea waves from a numerical model (Boussinesq) ;
- Shallow water and variable bathymetry

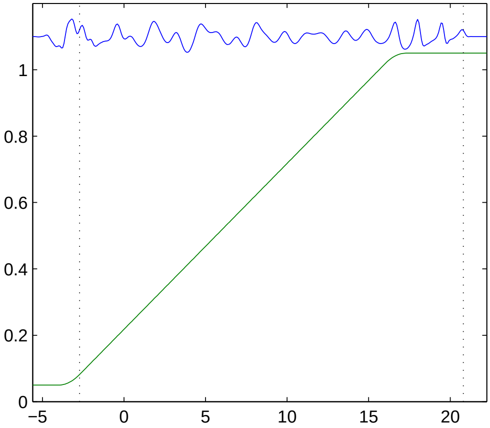


FIGURE : Example of simulated data (space domain) and bathymetry.

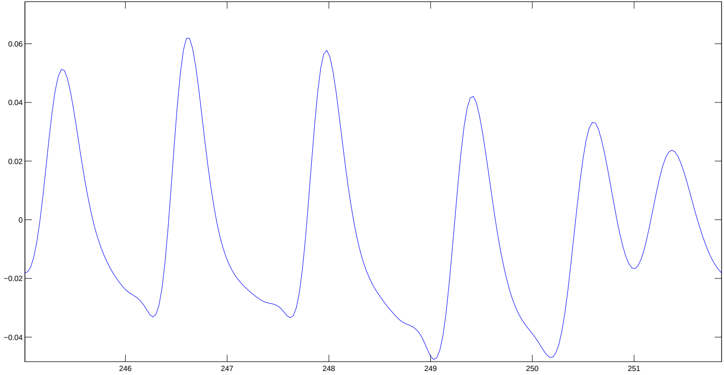


FIGURE : Extract for time series.

Parameter	Symmetric kernel	Proposed kernel
δ	-0.34	-0.16
μ	0.05	0.11
σ	0	0
ν	6.96	1.52
$\rho_{3,0} = -0.87$	0.00	-0.14
$\rho_{3,1} = 0.16$	0.15	0.16

TABLE : Parameters estimated from the data.

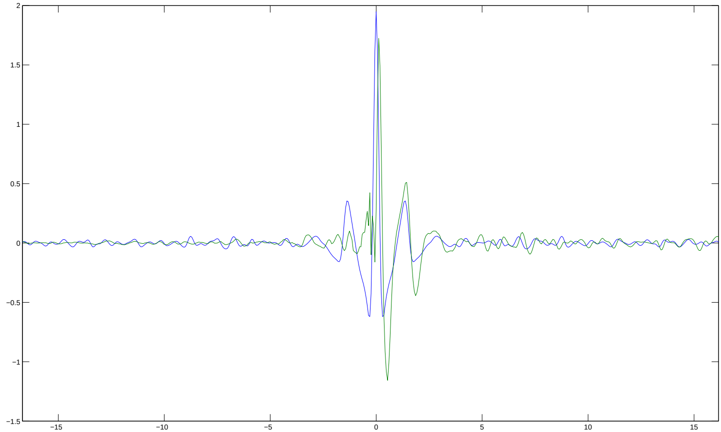


FIGURE : Estimated kernels. Blue : Symmetric ; Green : kernel with estimated phase.

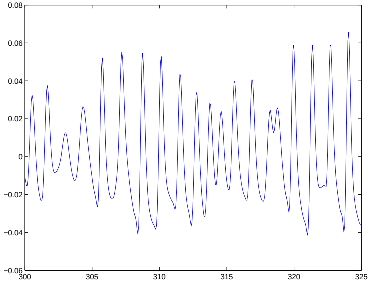


FIGURE : Observed data.

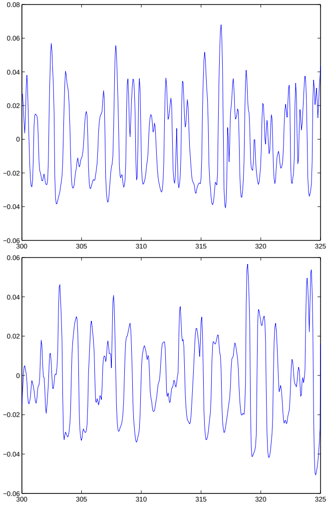
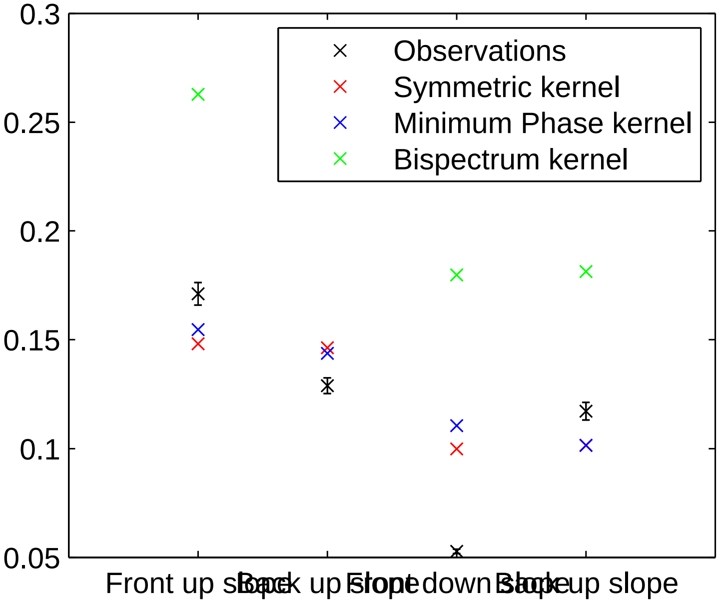


FIGURE : Simulated data with symmetric (top) and proposed kernel


0.3
0.25
0.2
0.15
0.1
0.05
Observations
Symmetric kernel
Minimum Phase kernel
Bispectrum kernel
Front up slope
Back up slope
Front down slope
Back up slope

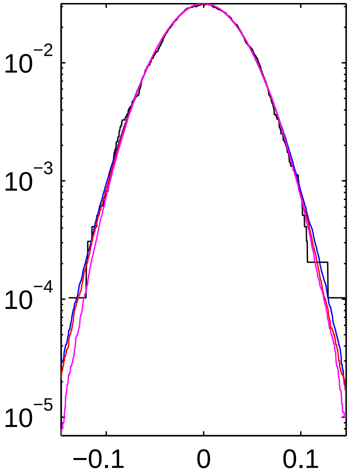


FIGURE : Up-crossing intensity at x=-2.72 (no depth effect). Empirical, Gaussian process and $\mathcal{LMA}$ with

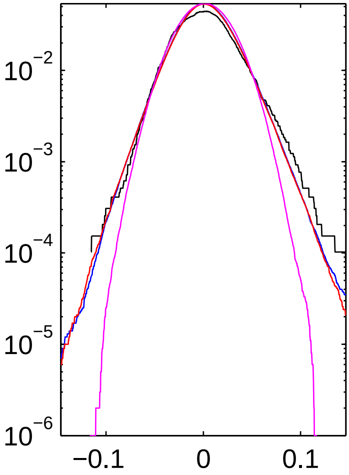


FIGURE : Up-crossing intensity at x=20.8 (important depth effect). Empirical, Gaussian process and $\mathcal{LMA}$

Plan

Conclusion - Perspectives

$\mathcal{LMA}$ are very versatile processes, able to describe asymmetries in the records, however there are still some work in process :

- The phase estimation is very instable, especially when the skewness is low ;
- Estimation of the phase thanks to splines ;
- Alternate scheme for marginal parameters estimation (PWM,EM...) ;
- Consequences on structural safety.

Thomas Galtier.
Note on the estimation of crossing intensity for laplace moving average.
Extremes, 14 :157–166, 2011.
10.1007/s10687-010-0116-4.

K. Podgórski and J. Wegener.
Non-gaussian fields with vertical and horizontal asymmetries.
Preprint, 2010.

K. Podgórski and J. Wegener.
Estimation for stochastic models driven by laplace motion.
Communications in statistics. Theory and methods, 40, 2011.

J. Wegener and K. Podgórski.
Stochastic fields with embedded shallow water dynamics.
Preprint, 2010.