ESTIMATES OF THE NUMBER OF RATIONAL MAPPINGS FROM A FIXED VARIETY TO VARIETIES OF GENERAL TYPE

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First we find effective bounds for the number of dominant rational maps $f : X \to Y$ between two fixed smooth projective varieties with ample canonical bundles. The bounds are of the type $\{A \cdot K_X^n\}^{\{B \cdot K_X^n\}^2}$, where $n = dimX$, K_X is the canonical bundle of X and A, B are some constants, depending only on n.

Then we show that for any variety X there exist numbers $c(X)$ and $C(X)$ with the following properties:

For any threefold Y of general type the number of dominant rational maps $f : X \to Y$ is bounded above by $c(X)$.

The number of threefolds Y, modulo birational equivalence, for which there exist dominant rational maps $f : X \to Y$, is bounded above by $C(X)$.
If, moreover, X is a threefold of general type, we prove that $c(X)$ and $C(X)$ only depend on the index r_{X_c} of the canonical model X_c of X and on $K^3_{X_c}$.

0. Introduction.

Let X and Y be algebraic varieties, i.e. complete integral schemes over a field of characteristic zero. Denote by $R(X, Y)$ the set of dominant rational maps $f : X \to Y$. Then the classical theorems of de Franchis [Fra] and Severi (cf. [Sam]) can be stated as follows:

Theorem 0.1.

a) (de Franchis): For any Riemann surface X and any hyperbolic Riemann surface Y the set $R(X, Y)$ is finite. Furthermore, there exists an upper bound for $\#R(X, Y)$ only in terms of X.

b) (Severi) For a fixed algebraic variety X there exist only finitely many hyperbolic Riemann surfaces Y such that $R(X, Y)$ is nonempty.

S. Kobayashi and T. Ochiai [Kob-Och] prove the following generalization of the de Franchis Theorem: If X is a Moishezon space and Y a compact complex space of general type, then the set of surjective meromorphic maps from X to Y

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is finite. Other generalizations can be found in [Des-Men1], [Nog], [Suz], and in the survey [Zai-Lin]. Generalizations of the second part of de Franchis' Theorem were given in [Ban1], [Ban2] and [Ban-Mar]. In the latter paper it is proved that for complex projective varieties X and Y with only canonical singularities and nef and big canonical classes K_X and K_Y respectively the number $\#R(X,Y)$ can be bounded in terms of the selfintersection K_X^3 of the canonical class of X and of the indices of X and Y. This bound is not effective. Effective bounds are known only if the varieties Y are curves or surfaces ([Kan], [How-Som2], [Tsa3]).

Section 1 of this paper contains an effective estimate of the number of mappings in $R(X,Y)$, provided that both varieties X and Y are smooth projective with ample canonical bundles K_X, K_Y respectively. This bound has the form $\{A \cdot K_X^n\}^{\{B \cdot K_X^n\}^2}$, where $n = dim X$, K_X is the canonical bundle of X and A, B are some constants, depending only on n.

This bound seems to be very big. But it is known that the bound cannot be polynomial in K_X^n ([Kan]). Moreover, even for the case of curves of genus 5 the best bound in [Kan] is of order $exp(30)$.

The idea to obtain this bound was used in ([How-Som1]) for proving finiteness of the automorphism group of a projective variety with ample canonical bundle. It could not be made effective at that time, as no effective variants of the Big Matsusaka theorem were available. Moreover, exponential bounds for the number of automorphisms are not interesting, as they should be linear in K_X^n ([Sza]).

In sections 2, 3 and 4 we generalize Severi's result (Theorem 0.1(b)) to higher dimensions.

Denote by $\mathcal{F}(X)$ the set of pairs (Y, f), where Y is of general type and $f \in R(X,Y)$. Let $\mathcal{F}_m(X) \subset \mathcal{F}(X)$ the subset of those pairs (Y, f) for which the m-th pluricanonical mapping of a desingularization of Y is birational onto its image. Consider the equivalence relations on $\mathcal{F}$ and $\mathcal{F}_m$: $(f : X \to Y) \sim (f_1 : X \to Y_1)$ iff $b \circ f = f_1$, where $b \in \mathrm{Bir}(Y, Y_1)$. The elements of $\mathcal{F}(X)/\sim$ we call *targets*.

The following conjecture is stated by Maehara ([Mae3]) as Iitaka's Conjecture based on Severi's Theorem:

Conjecture 0.2. *The set $\mathcal{F}(X)/\sim$ of targets is a finite set.*

Maehara proved in Proposition 6.5 in [Mae2] that in characteristic zero $\mathcal{F}_m(X)/\sim$ is finite for all m. In particular the Conjecture is valid for surfaces Y (take $m = 5$). Special cases and related aspects are discussed in [Tsa1] - [Tsa-3], [Des-Men2], [Des-Men3], [Mae1] and [How-Som2].

In section 3 we prove Conjecture 0.2 for the case that the targets are complex threefolds (Theorem 3.1).

For the proof we use the following theorem of Luo [Luo1], [Det]:

Theorem 0.3. *Consider the set of smooth threefolds Y of general type, and denote by $\chi(Y, \mathcal{O}_Y)$ the holomorphic Euler characteristic. Then for any fixed $\chi = \chi(Y, \mathcal{O}_Y)$, there is a universal integer m' such that $h^0(Y, \mathcal{O}_Y(m'K_Y)) \geq 2$. Furthermore, there is a universal integer m such that the m-th pluricanonical map $\Phi_{mK} : Y \to \Phi_{mK}(Y)$ maps birationally onto its image.*

In section 4 the domain is a threefold of general type. In this case we show (Theorem 4.1) that there is a bound for the number of targets $\#\mathcal{F}(X)/\sim$, which depends only on the selfintersection $K^3_{X_c}$ and the index r_{X_c} of the canonical model X_c of X.

The proof is based on the fact, due to Kollar [Kol1], that canonical threefolds with fixed Hilbert polynomial form a bounded family. Using semicontinuity theorems for the dimensions of cohomology groups we get estimates of the holomorphic Euler characteristics of the targets. Then we show that the graphs of maps under consideration, which map from canonical threefolds X with fixed index r_X and fixed K^3_X, form a finite number of algebraic families. The number of targets is bounded by the number of irreducible components of the members of these families.

In Section 5 we return to generalizations of de Franchis' result (Theorem 0.1(a)). Consider a threefold X of general type. We prove (Theorem 5.1) that there exists a bound for $\#R(X,Y)$, depending only on X. Namely, it depends on the selfintersection $K^3_{X_c}$ and on the index r_{X_c} of the canonical model X_c of X.

In the review of Sh. Kobayashi ([Kob], problem D3) the question is raised if for a compact complex space X and a hyperbolic compact complex space Y the number of surjective meromorphic maps from X to Y can be bounded only in terms of X. Theorem 5.1 is an answer to this question for threefolds of general type.

Further on all the varieties are complex; we do not make difference between line bundles, divisor classes and the divisors themselves, if no confusion may arise. We fix resp. recall the following notations, which are used in the paper:

X, Y – complex varieties;

$R(X,Y)$ – the set of rational dominant maps from X to Y;

$\mathcal{F}(X)$ – the set of pairs (Y,f), where Y is of general type and $f \in R(X,Y)$;

$\mathcal{F}_m(X)$ – the subset of those pairs (Y,f) for which the m-th pluricanonical mapping of a desingularization of Y is birational onto its image;

$(f : X \to Y) \sim (f_1 : X \to Y_1)$ iff $b \circ f = f_1$, where $b \in$Bir(Y,Y_1);

K_X – the canonical sheaf of a variety X with at most canonical singularities;

K^n_X – the n-times selfintersection of the class K_X, where $n = \dim X$;

$c_i(X)$ – the i^{th} Chern class of the variety X;

$H^i(X,D) = H^i(X,\mathcal{O}_X(D))$; $h^i(X,D) = \dim H^i(X,D)$;

$\chi(X,D) = \sum_{i=1}^n (-1)^i h^i(X,D)$;

X_c – the canonical model of a variety X of general type of $\dim_{\mathbb{C}} X \leq 3$;

r_X – the index of a variety X with at most canonical singularities;

Φ_{mK_Y} – the m-th pluricanonical map from a variety Y with at most canonical singularities.

1. Effective estimates of R(X,Y) for smooth manifolds X, Y with ample canonical bundles.

The main Theorem of this section is Theorem 1.6 below. It provides an effective estimate for $\#R(X,Y)$ if X,Y are smooth manifolds with ample canonical divisors.

We first recall some notations and facts about duality:

a) A subspace $E \subset \mathbb{P}^N$ is called linear if it is the projectivization of a linear subspace $E^a \subset \mathbb{C}^{N+1}$. Let $\rho : \mathbb{C}^{N+1} \to (\mathbb{C}^{N+1})^*$ denote the canonical isomorphism between $\mathbb{C}^{N+1}$ and the space $(\mathbb{C}^{N+1})^*$ of linear functionals on it, which is given by the standard hermitian product on $\mathbb{C}^{N+1}$. We denote by $(\mathbb{P}^N)^*$ resp. E^* the

projectivizations of $(\mathbb{C}^{N+1})^*$ resp. of $\rho(E^a) \subset (\mathbb{C}^{N+1})^*$, and call them the conjugate spaces to $\mathbb{P}^N$ resp. E. (We don't use the word 'dual' here in order not to have confusion with the notion of a dual variety which is defined below.)
b) We call a rational mapping $L\colon \mathbb{P}^N \to \mathbb{P}^M$ linear if it is the projectivization of a linear map $L^a\colon \mathbb{C}^{N+1} \to \mathbb{C}^{M+1}$. The projectivization of the induced map $(L^a)^*\colon (\mathbb{C}^{M+1})^* \to (\mathbb{C}^{N+1})^*$ is denoted by L^* and called the dual map to L.
c) Let Z be an n-dimensional projective variety embedded into the projective space $\mathbb{P}^N$. In any non-singular point $z \in Z$ the projective tangent plane T_z is well defined. In the conjugate projective space $(\mathbb{P}^N)^*$ we consider the set Z_0^V of all points $y \in (\mathbb{P}^N)^*$ such that the corresponding hyperplane $H_y \subset \mathbb{P}^N$ contains the tangent plane T_z to some nonsingular point $z \in Z$. We define the dual variety Z^V of the variety Z to be the closure of Z_0^V in the Zarisky topology.
d) The dual varieties have the following fundamental properties ([Del-Kat]):

1) If Z is nonsingular and Kodaira dimension $k(Z) > -\infty$, then Z^V is irreducible and codim $Z^V = 1$;
2) Moreover, if L is a hyperplane section of $Z \subset \mathbb{P}^N$, the degree $\deg Z^V$ of the variety Z^V may be computed by the Chern classes:

$$\deg Z^V = \sum_{i=0}^{n} (-1)^{n+i}(1+i)c_1^i(L)c_{n-i}(Z), \tag{1}$$

where $c_1(L)$ is the first Chern class of the line bundle corresponding to L;
3) $Z^{VV} = Z$.

Let X and Y be two smooth projective n-dimensional varieties with Kodaira dimension bigger than infinity, and let E and F be very ample line bundles on X and Y respectively. Then the varieties X and Y are canonically embedded into the projectivizations of the conjugate spaces to $H^0(X,E)$ and to $H^0(Y,F)$ respectively, which we denote by $\mathbb{P}^N$ and $\mathbb{P}^M$.

Let $f\colon X \dashrightarrow Y$ be a rational dominant mapping, and let $\Psi : H^0(Y,F) \to H^0(X,E)$ be an injective linear map. We call f to be induced by the map Ψ if the projectivization of the dual map Ψ^*, restricted to X, is the map f. Denote by $R(X,E,Y,F)$ the set

$$R(X,E,Y,F) = \{f \in R(X,Y) : f \text{ is induced by an injective linear map}$$
$$\Psi\colon H^0(Y,F) \to H^0(X,E)\}$$

Proposition 1.1. *If the set $R(X,E,Y,F)$ is finite, we have:*

$$\#R(X,E,Y,F) \le m^{\psi(E)}$$

where

$$m = \sum_{i=0}^{n} (-1)^{n+i}(1+i)c_1^i(E)c_{n-i}(X),$$
$$\psi(E) = (h^0(X,E))^2 - 1.$$

Before we start with the proof of Proposition 1.1, we need two Lemmas.

Lemma 1.2. *Denote by G the set of all linear injections $A : (\mathbb{P}^M)^* \to (\mathbb{P}^N)^*$ such that $A(Y^V) \subset X^V$. Then G is an quasiprojective subset of $\mathbb{P}^K, K = (N+1)(M+1)-1$, of degree*

$$\deg G \le (\deg X^V)^K. \tag{2}$$

Proof of Lemma 1.2. Any element of G is defined by a $(N+1)(M+1)$ matrix A, and its components (a_{ij}) may be considered as its coordinates in the projective space $\mathbb{P}^K,\ K = (N+1)(M+1)-1$. Since codim $X^V = 1$, it is defined in $(\mathbb{P}^N)^*$ by a single equation $F(z_0, \dots, z_N) = 0$ with $\deg F = \deg X^V$. If $y \in Y^V$ we have $A(y) \in X^V$, and so $F(Ay) = 0$. For a fixed point y and a fixed polynomial F this is an equation for the coordinates a_{ij} in the space $\mathbb{P}^K$.

This means that for any finite sequence of points $y_1, ..., y_r, y_i \in Y^V$, the set G is contained in the algebraic set $G^{(r)}$, defined by equations

$$\begin{aligned} &F(Ay_1) = 0 \\ &\vdots \\ &F(Ay_r) = 0 \end{aligned}$$

in the space $\mathbb{P}^K$.

Choose any point $y_1 \in Y^V$. Suppose that $\overline{G} \neq G^{(1)}$, where $\overline{G}$ denotes the Zariski closure of G in $\mathbb{P}^K$. Then there exists a point y_2 such that for some $A \in G^{(1)}$

$$F(Ay_2) \neq 0.$$

Define the set $G^{(2)}$ by the pair y_1, y_2. It follows that $G^{(2)} \subset G^{(1)}$, and for some component C of $G^{(1)}$ all components of $G^{(2)}$ which lie in C (if there are any at all) are of smaller dimension than C. After performing a finite number of such steps we get a set $y_1, y_2, ..., y_r$, such that $G^{(r)} = \overline{G}$. Hence, G can be defined in $\mathbb{P}^K$ by equations of degree $degF = degX^V$ only. Now, the inequality

$$\deg G \le (degX^V)^K.$$

follows from the

Sublemma (the analogue of the Bezout Theorem).

Let $X \subset \mathbb{P}^n$ be an irreducible variety, $\dim X = i, \deg X = a$. Let $F_1, ...F_s$ be homogeneous polynomials of degree d and $X_s = \{z \in \mathbb{P}^n : F_1(z) = F_2(z) = ... = F_s(z) = 0\}$. Assume that $X \cap X_s = \bigcup_{j=1}^{N} B_j$ is a union of irreducible components B_j. Then

$$\deg(X \cap X_s) = \sum_j \deg B_j \le ad^i.$$

Proof of the Sublemma. We perform induction by $i = \dim X$. If $i = 1$, there are two possibilities:

1. $F_k \big|_X = 0$ for all $k = 0, ...s$; then $X = B_1, N = 1$, and $\deg B_1 = \deg X = a$.
2. $F_1 \big|_X \not\equiv 0$. Then $X \cap X_s \subset X \cap X_1$ is a finite number T of points and

$$T \leq \deg(X \cap X_1) \leq ad.$$

Assume that the fact is true for every $i < m$. If $F_k \big|_X = 0$ for all $k = 1, ..., s$, then $N = 1, X = B_1$ and $\deg B_1 = a$. If $F_s \big|_X \not\equiv 0$, then $X \cap \{F_s = 0\} = \bigcup A_q$ is a union of irreducible components A_q such that $n_q = \dim A_q < m$ and

$$\sum_q \deg A_q \leq ad$$

(see, for example, [Har], Th.7.,ch. 1). Let $A_q \cap X_{s-1} = \bigcup_r B_q^r$. Since

$$\bigcup_{q,r} B_q^r = \left(\bigcup_q A_q \right) \cap X_{s-1} = X \cap X_s = \bigcup_j B_j,$$

and all B_q^r and B_j are irreducible, we obtain that for any j there are numbers (q,r) such that $B_j = B_q^r$. Thus

$$\sum_j \deg B_j \leq \sum_{q,r} \deg B_q^r.$$

By induction assumption

$$\sum_r \deg B_q^r \leq \deg A_q \ d^{n_q} \leq \deg A_q \ d^{m-1}.$$

Summation over q provides the desired inequality:

$$\sum_j \deg B_j \leq \sum_q \sum_r \deg B_q^r \leq \sum_q \deg A_q d^{m-1} \leq d^{m-1} \sum_q \deg A_q \leq ad^m.$$

□

Let $G = \bigcup_i G_i$ be the decomposition of G in irreducible components G_i.

Lemma 1.3. *Suppose that the points $t_1, t_2 \in G_i$ define linear maps $A_j, \quad j = 1, 2$, which are dual to linear projections $A_j^* : \mathbb{P}^N \to \mathbb{P}^M$ satisfying $A_j^*(X) = Y$ (i.e. $f_j := A_j^*|_X \in R(X, E, Y, F)$). Then $f_1 = f_2$.*

Proof of Lemma 1.3. From now on we fix a basis in $\mathbb{P}^N$ and $\mathbb{P}^M$. Let $t \in G_i$ and let A_t be a linear embedding $A_t : (\mathbb{P}^M)^* \to (\mathbb{P}^N)^*$ corresponding to a point t.

Consider the following diagram:

$$\begin{array}{ccccccc}
X & \subset & \mathbb{P}^N & \simeq & (\mathbb{P}^N)^* & \supset & X^V \\
\downarrow \ A_t^* \big|_X & & \downarrow \ A_t^* & & \cup & & \\
X_t & \subset & L_t & \simeq & E_t & \supset & X^V \cap E_t \\
\vdots\!\!\vee \ \tau_t & & \downarrow \tau_t & & \uparrow \ A_t & & \\
Y & \subset & \mathbb{P}^M & \simeq & (\mathbb{P}^M)^* & \supset & Y^V
\end{array}$$

In this diagram:

1) A_t is a linear embedding of $(\mathbb{P}^M)^*$ into $(\mathbb{P}^N)^*$;
2) $E_t = A_t((\mathbb{P}^M)^*)$ is a linear subspace of $(\mathbb{P}^N)^*$, which is isomorphic to $(\mathbb{P}^M)^*$;
3) The dual map $A_t^* : \mathbb{P}^N \to \mathbb{P}^M$ is the composition of a projection of $\mathbb{P}^N$ onto the subspace $L_t \subset \mathbb{P}^N$, which is dual to E_t, and of an isomorphism $\tau_t : L_t \to \mathbb{P}^M$ (recall that we have chosen a basis in $\mathbb{P}^N$ and a dual basis in $(\mathbb{P}^N)^*$). The projection we again denote by A_t^*.
4) $X_t = A_t^*(X) \subset L_t$.

Now the linear map A_t induces a dominant rational map $f_t : X \to Y$, iff $\tau_t(X_t) = Y$.

In order to proceed with the proof, the following two claims are needed.

Claim 1. *Let $R_i(x,t)$, $i = 1, ..., l$, be a finite number of polynomials in the variable $x \in \mathbb{P}^K$ with coefficients which are polynomials in the variable $t \in T$, where T is an irreducible projective variety. Let $V_t = \{x \in \mathbb{P}^K : R_1(x,t) = ... = R_l(x,t) = 0\}$. Assume that for some point $t_0 \in T$,*

$$r(x,t_0) = rank\{\frac{\partial R_i}{\partial x_j}|_{(x,t_0)}\} = k$$

for any $x \in V_{t_0}$. Then the set of points $t \in T$, such that $r < k$ for some $x \in V_t$, is proper and closed in T.

Proof. Consider the sets $A = \{(x,t) \in \mathbb{P}^K \times T : r(x,t) < k\}$ and $B = \{(x,t) \in \mathbb{P}^K \times T : x \in V_t\}$. Since $A \cap B$ is Zariski closed in $\mathbb{P}^K \times T$, its projection to T is Zariski closed in T. Since there is at least one point t_0, which does not belong to the image of this projection, it has to be a proper Zariski closed subset. □

By the assumption G_i has two points t_1, t_2, defining the maps f_1,$f_2 \in R(X,E,Y,F)$.

Claim 2. *Let $T' \subset G_i$ be the set of points $t \in G_i$, for which the image X_t of the projection of X into L_t is smooth and of the same dimension as X. Then T' is Zarisky open and contains t_1, t_2.*

Proof. We apply Claim 1 with $T = G_i$, $\mathbb{P}^K = L_t$ and where the $R_i(x,t)$, $i = 1, ..., l$ are the resultants of the polynomial equations of X in $\mathbb{P}^N$. Then $V_t = X_t$ is smooth in the point x iff the rank $r(x,t)$ is maximal. Since t_1, t_2 define the maps $f_1, f_2 \in R(X,E,Y,F)$, the varieties X_{t_i} are isomorphic to Y through the maps τ_{t_i}. Especially X_{t_1}, X_{t_2} are smooth and of the same dimension as X. Now Claim 2 follows from Claim 1. □

Using Claim 2, we will show that all $t \in T'$ correspond to maps $f_t \in R(X,E,Y,F)$, and moreover, that f_t does not depend on the parameter $t \in T'$. Especially, we get $f_{t_1} = f_{t_2}$.

Since X_{t_1} is isomorphic to Y, we get for the Kodaira dimensions $k(X_{t_1}) = k(Y) > -\infty$. Applying the invariance of plurigenera to the algebraic family with base T' and fiber X_t ([Har], 9.13, ch.3), we get that $k(X_t) > -\infty$ for all $t \in T'$. Then X_t^V has to be irreducible hypersurface in E_t, contained in $X^V \cap E_t$. Thus $X^V \cap E_t$ contains an irreducible component C_t such that $C_t = X_t^V$, and $C_t^V = X_t^{VV} = X_t$. Let $B_{t,i}$ be other irreducible components of $X^V \cap E_t$. Then $B_{t,i}^V \subset X_t$. Since X_t is irreducible, we get $\dim B_{t,i}^V < \dim X_t$. Thus, the intersection $X^V \cap E_t$ is a union of irreducible components $B_{t,i}$ and C_t, such that

a) the components $B_{t,i} \subset E_t$ are dual to some subsets of X_t (actually, the image of singular points of the projection of X to X_t) of dimension less than $n = \dim X$;

b) the component $C_t \subset E_t$ is the only one which has n - dimensional dual, and $C_t^V = X_t$.

For any point $t \in T'$ the variety $A_t(Y^V)$ is isomorphic to Y^V. Thus $\{A_t(Y^V)\}^V$ is isomorphic to $Y^{VV} = Y$ and, hence, it is n-dimensional. On the other hand, $A_t(Y^V)$ is contained in $X^V \cap E_t$ and is irreducible and of the same dimension as $X^V \cap E_t$. Hence, $A_t(Y^V) = C_t = X_t^V$.

That means that A_t is an isomorphism between $(\mathbb{P}^M)^*$ and E_t such that $A_t(Y^V) = X_t^V \subset E_t$. Then the dual isomorphism $\tau_t = A_t^* \big|_{L_t} \colon L_t \to \mathbb{P}^M$ maps $(X_t)^{VV} = X_t$ onto $Y^{VV} = Y$ (see [How-Som1]). It follows that the map $f_t = \tau_t \circ A_t^* \big|_X$ belongs to $R(X, E, Y, F)$. Since the latter set is finite and the family of the projections A_t^*, which give $f_t : X \to Y$, varies continuously over T', the map f_t does not depend on t. □

Proof of Proposition 1.1. A mapping $f \in R(X, E, Y, F)$ is, by definition, induced by a linear injection $\Psi \colon H^0(Y, F) \to H^0(X, E)$. More precisely, if we denote by $\tilde{f} : \mathbb{P}^N \to \mathbb{P}^M$ the projectivization of the dual map Ψ^* to Ψ, then $f = \tilde{f}|_X$.

Using the surjectivity of the maps $f : X \to Y$ and $\tilde{f} : \mathbb{P}^N \to \mathbb{P}^M$, an easy computation yields that the dual map $\tilde{f}^* \colon (\mathbb{P}^M)^* \to (\mathbb{P}^N)^*$ maps Y^V into X^V.

By Lemma 1.3 the number of mappings in $R(X, E, Y, F)$ is, at most, the number of irreducible components in G, which obviously does not exceed the sum of degrees of the these components. Since the mappings in $R(X, E, Y, F)$ are induced by injections of $H^0(Y, F)$ into $H^0(X, E)$, we have

$$M = \dim h^0(Y, F) - 1 \leq \dim h^0(X, E) - 1 = N.$$

By the formula (1) we get, for $\deg X^V$:

$$\deg X^V = \sum_{i=0}^{n} (-1)^{n+i} (1+i) c_1^i(E) \cdot c_{n-i}(X).$$

To obtain the statement of Proposition 1.1 it suffices to insert these values into equation (2). □

Using Proposition 1.1 we obtain the following

Theorem 1.6. *Let X, Y be two smooth complex projective varieties with ample canonical bundles K_X and K_Y. Let $R(X, Y)$ be the set of dominant rational maps $f \colon X \to Y$, and let the divisors sK_X, sK_Y be very ample (for example s may be $2 + 12n^n$ (*[Dem]*)). Then*

$$\#R(X,Y) \leq \left[(-1)^n \sum_{i=0}^{n} (1+i) s^i c_1^i(X) c_{n-i}(X) \right]^{\left\{ K_X^n \left(\frac{s^n}{n!} - \frac{s^{n-1}}{2(n-1)!} + q_{n-2}(s) \right) \right\}^2},$$

where, for each n, $q_{n-2}(s)$ is a universal polynomial of degree $n - 2$.

Proof of Theorem 1.6. Let $E = sK_X$, $F = sK_Y$ and $f \in R(X, E, Y, F)$. If f^* here denotes the pull back of pluricanonical forms by the rational map f, we get, by

[Iit], Theorem 5.3, that $f^*\colon H^0(Y, msK_Y) \to H^0(X, msK_X)$ is an injective linear map for any $m \in \mathbb{N}$. (Since the divisors E and F are very ample, any rational map $f \in R(X,Y)$ is even regular ([Ban1]), but we don't need this fact here.) It is easy to see that the map f is induced by the linear map f^*. That is why $R(X,Y) = R(X, sK_X, Y, sK_Y)$. Since this set is finite ([Kob-Och]), we can apply Proposition 1.1:

$$\begin{aligned}\#R(X,Y) &= \#R(X, sK_X, Y, sK_Y) \\ &\le \left[\sum_{i=0}^{n}(-1)^{n+i}(1+i)s^i c_1^i(K_X)c_{n-i}(X)\right]^{h^0(X,sK_X)^2-1}\end{aligned}$$

In this expression we substitute $-c_1(K_X)$ by $c_1(X)$ (these numbers are equal). Further, by the Riemann-Roch Theorem and the Vanishing Theorem for ample line bundles

$$h^0(X, sK_X) = \chi(X, sK_X) = K_X^n\left[\frac{s^n}{n!} - \frac{s^{n-1}}{2(n-1)!}\right] + P(s),$$

where $P(s) = \sum_{i=0}^{n-2} \alpha_i s^i$ is a polynomial of degree $n-2$ in s, the coefficients of which are linear combinations of monomials of the form $c_I(X) = c_{i_1}(X)\dots c_{i_k}(X), i_1 + \dots + i_k = n$.

According to ([Ful-Laz], [Cat-Sch]), there exist universal constants D_I, depending only on n, such that

$$|c_I(X)| \le D_I K_X^n.$$

It follows, that there are other universal constants $\tilde{D}_i$, $i = 0, ..., n-2$, which depend only on n, such that

$$|P(s)| \le \sum_{i=0}^{n-2}|\alpha_i|s^i \le K_X^n \cdot \sum_{i=0}^{n-2}\tilde{D}_i \cdot s^i.$$

Hence, it is possible to choose

$$q_{n-2}(s) = \sum_{i=0}^{n-2}\tilde{D}_i s^i.$$

□

2. Effective estimates for pluricanonical embeddings for threefolds.

This section is motivated by the following

Question 2.1. *Let Y be a smooth projective manifold of dimension n which is of general type. Does there exist an integer m, depending only on n, such that the m-th pluricanonical map $\Phi_{mK_Y} : Y \to \Phi_{mK_Y}(Y)$ is birational onto its image?*

It is well known (cf. [B-P-V]) that for curves we can choose $m = 3$, and for surfaces we can choose $m = 5$. Luo conjectured in [Luo1], [Luo2] that for the case of threefolds the answer to the question should also be affirmative. In these two papers, he proves his conjecture in ‘almost all’ possible cases. Especially he shows Theorem 0.3 (cf. Theorem 5.1, Corollary 5.3 of [Luo1]).

When the second named author gave a proof of Conjecture 0.2 for threefolds (cf. [Det]) he was not aware of the papers [Luo1] and [Luo2] of Luo. So he independently gave a proof of Theorem 0.3, using however the same basic idea (apparently both proofs were motivated by the paper [Flet] of Fletcher). Since the proof given in [Det] seems to use the basic idea in a shorter way and, moreover, easily gives effective bounds, we want to include it here. More precisely we prove the following statement:

Theorem 2.2. *Let C be a positive integer. Define $R =$lcm$(2, 3, \dots, 26C - 1)$ and $m =$lcm$(4R + 3, 143C + 5)$. Let Y any smooth projective threefolds of general type for which $\chi(Y, \mathcal{O}_Y) \leq C$ holds. Then $\Phi_{mK_Y} : Y \to \Phi_{mK_Y}(Y)$ is birational onto its image.*

For the convenience of the reader and to fix further notations we recall some facts on which the proof is built.

We need the Plurigenus Formula due to Barlow, Fletcher and Reid (cf. [Flet], [Rei2], see also [Kol-Mor], p.666 for the last part):

Theorem 2.3. *Let Y be a projective threefold with only canonical singularities. Then*

$$\chi(Y, mK_Y) = \frac{1}{12}(2m-1)m(m-1)K_Y^3 - (2m-1)\chi(Y, \mathcal{O}_Y) + \sum_Q l(Q, m)$$

with

$$l(Q, m) = \sum_{k=1}^{m-1} \frac{\overline{bk}(r - \overline{bk})}{2r} = \frac{r^2 - 1}{12}(m - \overline{m}) + \sum_{k=1}^{\overline{m}-1} \frac{\overline{bk}(r - \overline{bk})}{2r}$$

Here the summation takes place over a basket of singularities Q of type $\frac{1}{r}(a, -a, 1)$. $\overline{j}$ denotes the smallest nonnegative residue of j modulo r, and b is chosen such that $\overline{ab} = 1$.
Furthermore,

$$\text{index } (Y) = \text{lcm}\{r = r(Q) : Q \in \text{basket}\}$$

Hanamura ([Han]) proves:

Theorem 2.4. *Let Y be a smooth projective threefold of general type, which has a minimal or canonical model of index r. Then for any $m \geq m_0$ the m-th pluricanonical map is birational onto its image, where*

$$m_0 = 4r + 5 \text{ for } 1 \leq r \leq 2$$

$$m_0 = 4r + 4 \text{ for } 3 \leq r \leq 5$$

$$m_0 = 4r + 3 \text{ for } r \geq 6$$

In the last step of the proof we use the following theorem of Kollar (Corollary 4.8 in [Kol2]):

Theorem 2.5. *Assume that for a smooth projective complex threefold Y of general type we have $h^0(Y, lK_Y) \geq 2$. Then the $(11l+5)$-th pluricanonical map is birational onto its image.*

For estimating from below the terms $l(Q,m)$ in the Plurigenus Formula, we need two Propositions due to Fletcher [Flet]. In these Propositions $[s]$ denotes the integral part of $s \in \mathbb{R}$.

Proposition 2.6.

$$l(\frac{1}{r}(1,-1,1),m) = \frac{\overline{m}(\overline{m}-1)(3r+1-2\overline{m})}{12r} + \frac{r^2-1}{12}[\frac{m}{r}]$$

Proposition 2.7. *For $\alpha, \beta \in \mathbb{Z}$ with $0 \leq \beta \leq \alpha$ and for all $m \leq [(\alpha+1)/2]$, the following holds:*

$$l(\frac{1}{\alpha}(a,-a,1),m) \geq l(\frac{1}{\beta}(1,-1,1),m)$$

The basic idea of the proof is the following: We look at the canonical model of the threefold Y, which exists by the famous result of Mori [Mor], combined with results of Fujita [Fuj], Benveniste [Ben] and Kawamata [Kaw]. If the index of the canonical model is small, we can finish the proof by using Hanamura's Theorem. If the index is big, we use the Plurigenus Formula due to Barlow, Fletcher and Reid to show that for some m we have $h^0(Y, mK_Y) \geq 2$, and finish the proof by using Kollar's theorem.

Proof of Theorem 2.2. We first observe that by a theorem due to Elkik [Elk] and Flenner [Flen] (cf. [Rei2], p.363), canonical singularities are rational singularities. Hence, by the degeneration of the Leray spectral sequence we have

$$\chi(Y, \mathcal{O}_Y) = \chi(Y_c, \mathcal{O}_{Y_c}).$$

If the index of Y_c divides R, we apply Hanamura's Theorem and get that $\Phi_{(4R+3)K_Y}$ embeds birationally. Hence, we may assume that the index does not divide R. Then in the Plurigenus Formula we necessarily have at least one singularity $\tilde{Q}$ in the basket of singularities which is of the type $\frac{1}{r}(a,-a,1)$ with $r \geq 26C$. Applying a vanishing theorem for ample sheaves (cf. Theorem 4.1 in [Flet]), the fact that $K_{Y_c}^3 > 0$ (since K_{Y_c} is an ample $\mathbb{Q}$-divisor) and finally the Propositions 2.6 and 2.7 of Fletcher, we obtain:

$$\begin{aligned}
&h^0(Y_c, (13C)K_{Y_c}) \\
&= \chi(Y_c, (13C)K_{Y_c}) \\
&\geq (1-26C)\chi(Y_c, \mathcal{O}_{Y_c}) + \sum_{Q \in \text{basket}} l(Q, 13C) \\
&\geq (1-26C)C + l(\tilde{Q}, 13C) \\
&\geq (1-26C)C + l(\frac{1}{26C}(1,-1,1), 13C) \\
&= (1-26C)C + \frac{13C(13C-1)(78C+1-26C)}{312C}
\end{aligned}$$

$$= \frac{52C^2 - 15C - 1}{24} \geq \frac{36}{24} = 1.5$$

The last inequality is true since $C \geq 1$. Since $h^0(Y_c, (13C)K_{Y_c})$ is an integer, it has to be at least 2. From the Definition of canonical singularities it easily follows (cf. e.g. [Rei1], p.277, [Rei2], p.355 or [Flet], p.225) that $h^0(Y, (13C)K_Y) \geq 2$. Now we can finish the proof by applying Theorem 2.5 due to Kollar. □

Despite the fact that our $m = m(C)$ is explicit, it is so huge that it is only of theoretical interest. For example for $C = 1$ one can choose $m = 269$ ([Flet]), but for $C = 1$ our m is already for of the size 10^{13}. Moreover, for all examples of threefolds of general type which are known so far, any $m \geq 7$ works. So we guess there should exist a bound which is independent of the size of the holomorphic Euler characteristic.

3. Iitaka-Severi's Conjecture for threefolds.

The claim of this section is the following

Theorem 3.1. *Let X be a fixed complex variety. Then the set of targets $\mathcal{F}(X)/\sim$ with $dim_{\mathbb{C}} Y \leq 3$ is a finite set.*

By Proposition 6.5 of Maehara [Mae2] it is sufficient to show the following: There exists a natural number m, only depending on X, such that $\mathcal{F}(X) \subset \mathcal{F}_m(X)$ for varieties Y with $\dim_{\mathbb{C}} Y \leq 3$. Since we prove finiteness only up to birational equivalence, we may assume, without loss of generality, that X and all Y in Theorem 3.1 are nonsingular projective varieties. This is by virtue of Hironaka's resolution theorem [Hir], cf. also [Uen], p.73. Hence, using Theorem 2.2 or Theorem 0.3 of Luo we get Theorem 3.1 as a consequence of the following:

Proposition 3.2. *Let X be a fixed smooth projective variety and $f : X \to Y$ a dominant rational map to another smooth projective variety Y with $dim_{\mathbb{C}} Y = n$. Then we have*

$$\chi(Y, \mathcal{O}_Y) \leq \sum_{\{i|2i \leq n\}} h^i(X, \mathcal{O}_X)$$

Proof of Proposition 3.3. First we obtain, by Hodge theory on compact Kähler manifolds (cf. [Gri-Har], or [Iit], p.199)

$$h^i(Y, \mathcal{O}_Y) = h^0(Y, \Omega^i_Y),$$

where $i = 1, ...n$. The same kind of equalities hold for X. Now by [Iit], Theorem 5.3, we obtain that

$$h^0(Y, \Omega^i_Y) \leq h^0(X, \Omega^i_X),$$

where again $i = 1, ..., n$. Hence, we can conclude:

$$\chi(Y, \mathcal{O}_Y) \leq \sum_{\{i|2i \leq n\}} h^i(Y, \mathcal{O}_Y) = \sum_{\{i|2i \leq n\}} h^0(Y, \Omega^i_Y) \leq$$

$$\leq \sum_{\{i|2i \leq n\}} h^0(X, \Omega^i_X) = \sum_{\{i|2i \leq n\}} h^i(X, \mathcal{O}_X)$$

□

4. On the number of Targets.

Let X be a smooth threefold of general type, and define $r = r_{X_c}$, $k = K^3_{X_c}$.

Theorem 4.1. *There exists a universal constant $C(r,k)$, depending only on r and k, such that*

$$\#(\mathcal{F}(X)/\sim) \leq C(r,k)$$

Theorem 4.2. *There exists a universal constant $C'(r,k)$, depending only on r and k, such that if Y is a smooth threefold and $R(X,Y) \neq \emptyset$, then*

$$r_{Y_c} \leq C'(r,k)$$

The rest of this section deals with the proof of Theorem 4.1 and Theorem 4.2, which we prove simultaneously. We fix positive integers r and k. Denote by $\mathbb{X}(r,k)$ the set of threefolds X_c with only canonical singularities and ample canonical sheaves K_{X_c}, which satisfy $r_{X_c} = r$, $K^3_{X_c} = k$. Let X be a smooth threefold such that $X_c \in \mathbb{X}(r,k)$.

a) In this part of the proof we only consider targets $((Y,f)/\sim) \in (\mathcal{F}(X)/\sim)$ with $\dim_{\mathbb{C}} Y = 3$.

Due to Theorem 2.4 of Hanamura, the map

$$\Phi_{9rK_X} : X \to \mathbb{P}^N$$

is birational onto its image, where, by ([Mat-Mum])

$$N = \dim H^0(X, 9rK_X) - 1 \leq 9^3 r^3 k + 3.$$

Moreover, by [Ban-Mar], Lemma 1 (cf. also Proposition 2, part 2), the degree d_X of the image $X' = \Phi_{9rK_X}(X)$ has the bound

$$d_X \leq 9^3 r^3 k.$$

Let Y be a smooth threefolds of general type with $R(X,Y) \neq \emptyset$.

Proposition 4.3. *There exists a universal constant $C_1(r,k)$, depending only on r and k, such that we have*

$$\chi(Y, \mathcal{O}_Y) \leq C_1(r,k)$$

Proof of Proposition 4.3. By Proposition 3.2 we have

$$\chi(Y, \mathcal{O}_Y) \leq h^2(X, \mathcal{O}_X) + 1.$$

In the Hilbert polynomials $\chi(X_c, mrK_{X_c})$ the expressions $l(Q, rm)$, cf. Theorem 2.3, are linear in m, and so the two highest coefficients of the polynomial $\chi(X_c, mrK_{X_c})$ in the variable m only depend on r^3k. But then by Theorem 2.1.3 of Kollar [Kol1], the family of the (X_c, rK_{X_c}), where $X_c \in \mathbb{X}(r,k)$, is a bounded family. That means there exists a morphism $\pi : \mathcal{X} \to S$ between (not necessarily complete) varieties $\mathcal{X}$ and S and a π-ample Cartier divisor D on $\mathcal{X}$ such that every

(X_c, rK_{X_c}) is isomorphic to $(\pi^{-1}(s), D|_{\pi^{-1}(s)})$ for some $s \in S$. So it is sufficient to prove that there exists a constant C_0 which satisfies: For all $s \in S$ and for some desingularization $X(s)$ of $\pi^{-1}(s)$ we have $h^2(X(s), \mathcal{O}_{X(s)}) \leq C_0$[1].

This is shown by using first generic uniform desingularization of the family $\pi : \mathcal{X} \to S$ (cf. [Hir], [Bin-Fle]), and afterwards a semi-continuity theorem (cf. [Gro], [Gra]): By applying generic uniform desingularization and induction on the dimension there exist finitely many subvarieties $S_i, i = 1, ..., l$, which cover S, and morphisms $\Psi_i : \mathcal{Y}_i \to S_i$ between varieties $\mathcal{Y}_i$ and S_i which desingularize $\mathcal{X}_i := \pi^{-1}(S_i)$ fiberwise, i.e., there exist morphisms $\Phi_i : \mathcal{Y}_i \to \mathcal{X}_i$ over S_i such that for any $s \in S_i$ the map $\Phi_i : \Psi_i^{-1}(s) \to \pi^{-1}(s)$ is a desingularization.

Using semi-continuity for the families $\Phi_i : \mathcal{Y}_i \to S_i$, we obtain finitely many subvarieties $S_{ij}, j = 1, ..., l_i$ of S_i, which cover S_i, and have the following property: If we denote $\mathcal{Y}_{ij} := \Phi_i^{-1}(S_{ij})$ and $\Phi_{ij} := \Phi_i|_{\mathcal{Y}_{ij}}$, we get that for the families $\Phi_{ij} : \mathcal{Y}_{ij} \to S_{ij}$ the number $C_{ij} := h^2(\Phi_{ij}^{-1}(s), \mathcal{O}_{\Phi_{ij}^{-1}(s)})$ is constant for $s \in S_{ij}$. Hence, $C_0 := \max_{i=1,...,l; j=1,...,l_i} C_{ij}$ has the desired property. □

Remark. Proposition 4.3 can also be proved as follows. By a result of Milnor ([Mil]) the Betti numbers of the variety $X' = \Phi_{9rK_X}(X)$ have estimates depending on its degree $d_X \leq 9^3 r^3 k$ only. From the standard exact cohomology sequences and dualities it easily follows that $h^{2,0}(X)$ may be estimated by Betti numbers of X'.

Using Proposition 4.3 and Theorem 2.2 we can choose an integer $p = p(r, k)$, such that p is divisible by r, $p \geq 9r$ and

$$\Phi_{pK_Y} : Y \to \mathbb{P}^M$$

is birational onto its image, where, by ([Mat-Mum])

$$M = \dim H^0(Y, pK_Y) - 1 \leq p^3 k + 3.$$

Lemma 4.4.

1) The degree of $Y' := \Phi_{pK_Y}(Y) \subset \mathbb{P}^M$ is smaller than $\deg X' \leq p^3 k$.

2) For any map $f \in R(X', Y')$ the degree d_f of its graph $\Gamma_f \subset \mathbb{P}^N \times \mathbb{P}^M$ is not greater than $8p^3 k$.

Lemma 4.4 is a particular case of part 2 and 3 of Proposition 2 of [Ban-Mar] for $n = 3$, applied to the threefolds X_c, Y_c and linear systems $|pK_{X_c}|, |pK_{Y_c}|$. We have to note only that Proposition 2 and Lemma 1 in [Ban-Mar] is stated for Cartier divisors. But only the fact that they are $\mathbb{Q}$-Cartier is used in their proofs. □

By Proposition 1 of the same paper ([Ban-Mar]), there exist algebraic families $(\mathcal{X}, p_X, T)$, $(\mathcal{Z}, p_Z, V)$, $(\mathcal{Y}, p_Y, U)$ with constructive bases and projections $\pi_U : V \to U, \pi_T : V \to T$, with the following properties:

1) For any $X_c \in \mathbb{X}(r, k)$, there is a point $t \in T$, such that X_c is birational to $X = p_X^{-1}(t)$, and all points $t \in T$ have this property.

2) For any Y with $R(X, Y) \neq \emptyset$ for some $X_c \in \mathbb{X}(r, k)$, there is a point $u \in U$, such that Y is birational to $Y = p_Y^{-1}(u)$, and all points $u \in U$ have this property.

[1]Remark that by an easy argument like in the proof of Proposition 3.2 any two such desingularizations have the same $h^2(X(s), \mathcal{O}_{X(s)})$.

3) For any dominant rational map $f : p_X^{-1}(t) = X \to Y = p_Y^{-1}(u)$, there is a point $v \in V$, such that $\pi_U(v) = u$, $\pi_T(v) = t$, $p_V^{-1}(v)$ is a graph of the map f, and all points $v \in V$ have this property.

Let $\tilde{V} = \{(t, v) \in T \times V | \pi_T(v) = t\}$, and denote by p_T resp. p_V the projections to the first resp. to the second factor. Through the composed map $\pi_U \circ p_V : \tilde{V} \to U$ the variety $\tilde{V}$ is also a variety over U. Let $\tilde{\mathcal{Y}} = \tilde{V} \times_U \mathcal{Y}$ be obtained by base change, and denote the projection to the first factor by $p_{\tilde{V}} : \tilde{\mathcal{Y}} \to \tilde{V}$. Then we have

$$\tilde{\mathcal{Y}} \quad \xrightarrow{p_{\tilde{V}}} \quad \tilde{V} \quad \xrightarrow{p_T} \quad T.$$

In this diagram, for every $t \in T$, the set $p_T^{-1}(t)$ can be considered as the set of graphs of dominant rational maps $f : X \to Y$, where $X = p_X^{-1}(t)$, and $p_{\tilde{V}} : \tilde{\mathcal{Y}} \to \tilde{V}$ is the universal family of threefolds Y over the graphs of $f : X \to Y$.

By applying the process of local uniform desingularization, described in Proposition 4.3, to the family $p_{\tilde{V}} : \tilde{\mathcal{Y}} \to \tilde{V}$, we obtain a finite number of smooth families $(p_{\tilde{V}})_i : (\tilde{\mathcal{Y}})_i \to (\tilde{V})_i$, $i = 1, ..., l$, the bases $(\tilde{V})_i$ of which are connected and cover $\tilde{V}$, and the fibers of which are desingularizations of the fibers of $p_{\tilde{V}} : \tilde{\mathcal{Y}} \to \tilde{V}$. For any i the map $(p_{\tilde{V}})_i : (\tilde{\mathcal{Y}})_i \to (\tilde{V})_i$ is a smooth family of projective threefolds of general type over a connected base $(\tilde{V})_i$. By a theorem of J.Kollar and Sh.Mori ([Kol-Mor], Theorem 12.7.6.2) there is an algebraic map ϕ_i from $(\tilde{V})_i$ to the birational equivalence classes of the fibers of $(p_{\tilde{V}})_i : (\tilde{\mathcal{Y}})_i \to (\tilde{V})_i$. Moreover, all these fibers have the same Hilbert function.

From this fact two conclusions can be derived:

1. Since the index of a canonical threefold can be bounded in terms of the Hilbert function ([Kol-Mor], p.666), the indices of the canonical models of the fibers of the family $(p_{\tilde{V}})_i : (\tilde{\mathcal{Y}})_i \to (\tilde{V})_i$ vary in a finite set of natural numbers, only.

2. Let $(p_T)_i := p_T|_{(\tilde{V})_i}$ and $n_i(t)$ be the number of irreducible components of $(p_T)_i^{-1}(t)$ (it may be zero). Define, for $X = p_X^{-1}(t)$, $\mathcal{G}(X) = \{Y | (Y, f) \in \mathcal{F}(X)\}$, and let $\sim$ denote birational equivalence on $\mathcal{G}(X)$. Since $(\#\mathcal{G}(X)/\sim) < \infty$, it follows that the restriction ϕ_i to $(p_T)_i^{-1}(t)$ has to be constant on the connected components of $(p_T)_i^{-1}(t)$. Then

$$\#(\mathcal{G}(X)/\sim) \leq \sum_{i=1}^{l} n_i(t).$$

Since from the beginning the constructions of all the families were algebraic and defined only by the constants r and k, we have proved Theorem 4.2, and also the following

Lemma 4.5. *There exists a universal constant $C_2(r, k)$, depending only on r and k, such that we have*

$$(\#\mathcal{G}(X)/\sim) \leq C_2(r, k).$$

Next, we look at the map

$$(\pi_T, \pi_U) : V \to T \times U.$$

It is algebraic and any point of the fiber over $(t, u) \in T \times U$ defines a map from $R(\pi_T^{-1}(t), \pi_U^{-1}(u))$. The last set is finite for all (t, u), so we get:

Lemma 4.6. *There exists a universal constant $C_3(r,k)$, depending only on r and k, such that*

$$\#R(X,Y) \leq C_3(r,k).$$

From Lemma 4.5 and Lemma 4.6 the statement of Theorem 4.1 for 3-dimensional targets is immediate. The desired bound may be chosen as $C_2(r,k)C_3(r,k)$. □

b) Now we consider targets $((Y,f)/\sim) \in (\mathcal{F}(X)/\sim)$ with $\dim_{\mathbb{C}} Y \leq 2$. For these targets we know that the indices of the Y_c are 1 or 2. So we can repeat the same argument as above, omitting however Proposition 4.3. The only change which has to be done is replacing the moduli spaces due to [Kol-Mor] by the respective moduli spaces for surfaces or curves. So we get Theorem 4.1, and in particular Lemma 4.5 and Lemma 4.6, also for these kinds of targets. □

Remark. According to [Ban-Mar], there exists a universal function σ in two variables, such that $\#R(X,Y) \leq \sigma(r \cdot r_{Y_c}, k)$. This fact, together with Theorem 4.2, yields an alternative proof of Lemma 4.6.

5. A Conjecture of Kobayashi for threefolds of general type.

In this section we prove

Theorem 5.1. *For any complex variety X there is a number $c(X)$ such that*

$$\#R(X,Y) \leq c(X)$$

for any complex variety Y of general type with $\dim_{\mathbb{C}} Y \leq 3$. If X is a threefold of general type, then $c(X)$ can be expressed only in terms of r_{X_c} and $K^3_{X_c}$.

Proof of Theorem 5.1. Like in section 3 we may assume that X and Y are smooth projective varieties. By [Kob-Och], $\#R(X,Y)$ is finite for every fixed Y. By Theorem 3.1, we know that for given X there exist only finitely many such Y, up to birational equivalence. Since birational equivalence does not effect the number $\#R(X,Y)$, the first statement follows.

Let X now be a projective threefold of general type. Then the second statement is just Lemma 4.6. □

Remark 5.2. *The estimate which is given in Theorem 5.1 is not effective.*

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