Deformation of Compact Riemann Surfaces with Distinguished Points

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Abstract: We show that the family of the universal coverings of the fibers of a real analytic family $\pi : X \to B \subset \mathbb{C}^n$ of punctured compact Riemann surfaces is, locally w.r.t. B, bianalytically and on the fibers holomorphically a cartesian product.

1 Introduction and Statement of the Results

The talk which I gave at the workshop 'Complex Analysis' at Wuppertal was essentially about the main theorem of the article [DG], which contains some joint work with H. Grauert and is one part of a planned much bigger project. In this project we want to try to prove the conjecture (cf. [Z]) that the set of all complex curves $\Gamma_d \subset \mathbb{P}^2$ of degree d for which $\mathbb{P}^2 \backslash \Gamma_d$ is hyperbolic in the sense of Kobayashi (cf. [K]), contains, for every $d \geq 5$, a Zariski open set. The statement which we will need for this purpose is Corollary 1.3 below.

Let us first define the objects which we want to deal with:
Assume that X, B are real analytic manifolds and that $\pi : X \to B$ is a surjective, proper, smooth real analytic map.
Further assume that there exists a special atlas $\mathcal{A}$ of charts of X of the following form:

$$(w_1, w_2, t_1, \ldots, t_{2n}) : U\,(\subset X) \to \mathbb{R}^{2n+2}$$

s.th. for every $x \in U$ the $(t_1, \ldots, t_{2n})$ are real analytic coordinates around $\pi(x) \in B$; and s.th. for two such charts the w–components depend holomorphically on each other for fixed t–components.
The atlas $\mathcal{A}$ makes every fiber $X_t := \pi^{-1}(t), t \in B$ a compact complex submanifold of X. We assume that every X_t is a Riemann surface of genus p. Moreover we assume that there are given real analytic cross sections $s_1(t), \ldots, s_m(t)$ in X over B s.th. for every $t \in B$ the points $s_1(t), \ldots, s_m(t)$ are disjoint.

Definition 1.1 *X is called a real analytic deformation of Riemann surfaces of genus p with m distinguished points. If additionally X, B are complex manifolds, $\mathcal{A}$ is a holomorphic atlas, and $\pi : X \to B$ and $s_1, \ldots, s_m$ are holomorphic, then X is called a holomorphic deformation of Riemann surfaces of genus p with m distinguished points.*

In all what follows we mean by real analytic resp. holomorphic deformations these special kinds of deformations.
What we want to prove is the following:

Theorem 1.2 *Let $\pi : X \to B$ be a real analytic deformation of compact Riemann surfaces of genus p with m distinguished cross sections $s_1, \ldots, s_m : B \to X$.*
Let $X' = X \setminus \{s_1(B), \ldots, s_m(B)\}$ and denote by $\hat{X} \to B$ the family of the universal coverings of the fibers of $X' \to B$.
Then for every $t_0 \in B$ there exists a neighborhood $U = U(t_0) \subset B$ such that the following

is true: There exist finitely many automorphisms of $\hat{X}_{t_0}$ with coefficients depending real analytically on $t \in U$, which, for fixed $t \in B$, generate a subgroup Γ_t of the automorphism group of $\hat{X}_{t_0}$ which acts properly discontinuously on $\hat{X}_{t_0}$. There further exists a fiber-preserving real bianalytic and on the fibers biholomorphic map

$$F : X'|U \to (\hat{X}_{t_0}/\Gamma_t)_{t\in U}.$$

Especially there exists a fiber-preserving real bianalytic and on the fibers biholomorphic map

$$\hat{F} : \hat{X}|_U \to \hat{X}_{t_0} \times U$$

i.e. the family $\hat{X} \to B$ is locally a cartesian product.

Corollary 1.3 *If the fibers of the family $X' \to B$ are hyperbolic Riemann surfaces, i.e. if $2p+m-3 \geq 0$, then the family of the Poincare metrics on the fibers depends real analytic on the fiber parameter.*

For holomorphic deformations (of our kind) the situation is completely different from that of Theorem 1 2, since we have:

Corollary 1.4 *Let $\pi : X \to B$ be a holomorphic deformation of compact Riemann surfaces of genus p with m distinguished cross sections $s_1, \ldots, s_m : B \to X$. Assume that the family $\hat{X} \to B$ of the universal coverings of the family $(X \setminus \{s_1(B), \ldots, s_m(B)\}) \to B$ has the following property: For every $t_0 \in B$ there exists a neighborhood $U = U(t_0) \subset B$ and a fiber-preserving biholomorphic map*

$$F : \hat{X}|_U \to \hat{X}_{t_0} \times U$$

i.e the family $\hat{X} \to B$ is locally a cartesian product.
Then the deformation $\pi : X \to B$ itself is locally a cartesian product (in the same sense) except the case of non-punctured tori ($p = 1, m = 0$). In the case of non-punctured tori the assertion actually is wrong.

The proof of Corollary 1.4 uses that under the assumptions of Theorem 1.4 the coefficients of our special system of generators for the covering transformation groups (cf. Theorem 1.2) are real valued and depend holomorphically on $t \in U$, and hence are constant. The proof of Theorem 1.2 (which already is given, for the most important case ($2p+m-3 \geq 0$), in [DG]) consists mainly of two steps: In sect 2 we develop some deformation theory for real analytic deformations (of our kind), which is very similar to the classical deformation theory of compact complex manifolds, cf. [KS1]. [KS2], [KNS]. Especially we show that there exist real analytic semiuniversal deformations for real analytic deformations. In sect. 3 we then will construct, for a given punctured compact Riemann surface Y', one special real analytic deformation with bijective infinitesimal deformation (cf. sect.2) by deviding a cartesian product fiberwise through a real analytic deformation of the group of the covering transformations of the universal covering of Y'. Then the family of the universal coverings is, by construction, a cartesian product, and from the general deformation theory in sect. 2 it will follow that locally an arbitrary real analytic deformation can be obtained by lifting from the special real analytic deformations constructed in sect. 3, thus proving Theorem 1.2.
The statement of Corollary 1.3 can probably be proved more directly by constructing the family of the Poincare metrics on the fibers by continuity method, but we didn't

work that out. The statement of Theorem 1.2 can also be obtained by proving that the so called Fricke coordinates of a (at least real analytic) family of punctured Riemann surfaces (see [A]) depend real analytically on the fiber parameter. This result probably was known before by completely different methods, using quasi conformal mappings and the Teichmüller theory (cf. [A], he uses the result that the Fricke coordinates are real analytic coordinates for the Teichmüller space without proof). From this result our results can be obtained quite easily by the theory of Teichmüller spaces.

2 Existence of Semiuniversal Deformation

First we define the infinitesimal deformation of real analytic deformations (cf. Definition 1.1) in a point $t \in B$. Let Ψ_t be the sheaf of holomorphic cross sections in the tangent bundle of X with respect to the mixed real analytic and holomorphic structure, restricted to the fiber X_t, and Θ_t^* be the sheaf of holomorphic cross sections in the holomorphic tangent bundle of X_t which are zero in $s_1(t), \ldots, s_m(t)$. We take a covering $\mathcal{U} = \{U_1, \ldots, U_l\}$ of open subsets $U_1, \ldots, U_l$ in X_t s.th. for every $\xi \in T_{B,t}$, there exist fields $\eta_\nu \in H^0(U_\nu, \Psi_t|_{U_\nu})$ for which $\pi_*(\eta_\nu) \equiv \xi$ and η_ν is tangent along the $s_1(t), \ldots, s_m(t)$. Then the η_ν, $\nu = 1, \ldots, l$ define a 0-cochain $\{\eta_\nu\} \in C^0(\mathcal{U}, \Psi_t)$ in the Čech complex defined by the sections of the sheaf Ψ_t and the covering $\mathcal{U}$. It yields a cocycle $d\{\eta_\nu\} \in Z^1(\mathcal{U}, \Theta_t^*)$ which, by passing to the limit, yields a cohomology class $\rho_t(\xi) \in H^1(X_t, \Theta_t^*)$. This is uniquely determined and $\rho_t : T_{B,t} \to H^1(X_t, \Theta_t^*)$ is a real linear map.
If the deformation is holomorphic then we define completely analogously using the holomorphic instead of the mixed real analytic and holomorphic structure. We get a complex linear map $\rho_t : T_{B,t} \to H^1(X_t, \Theta_t^*)$.

Definition 2.1 *The map $\rho_t : T_{B,t} \to H^1(X_t, \Theta_t^*)$ is called the infinitesimal deformation of the real analytic resp. holomorphic deformation in the point $t \in B$.*

We have the following

Proposition 2.2 *A real analytic resp. holomorphic deformation is stable in $t \in B$ with resp. to $s_1, \ldots, s_m$ (i.e. there exist some charts out of our special atlas $\mathcal{A}$ covering X_t s.th. in them the deformation is a cartesian product over the double point in t) iff $\rho_t(\xi) \equiv 0$ in $T_{B,t}$.*

The proof goes along the same lines as that of the corresponding result in [KS1].

Definition 2.3 *A real analytic (resp. holomorphic) deformation $\pi : X \to B$ is called semiuniversal in $t \in B$ if:*
1) If $\psi : Z \to G$ is an other real analytic (resp. holomorphic) deformation with $0 \in G$ and $Z_0 = X_t$ then there is a real analytic (resp. holomorphic) map $\alpha : U(0) \to B$ with $\alpha(0) = t$ s.th. $X \circ \alpha = X \times_\alpha U$ is isomorphic to $Z|_U$ under an isomorphism which is on $Z_0 = X_t$ the identity.
2) The total derivative $d\alpha : T_{G,0} \to T_{B,t}$ is uniquely determined.

Proposition 2.4 *If Y is a compact Riemann surface with distinguished points $P_1, \ldots, P_m$ in Y there exists a holomorphic semiuniversal deformation $\pi : X \to B \subset \mathbb{C}^n$ with $Y = X_0$, $P_i = s_i(0), i = 1, \ldots, m$ and $n = \dim_{\mathbb{C}} H^1(X_0, \Theta_0^*)$.*

For a proof cf. [DG].

Proposition 2.5 *If $\pi : X \to B$ is semiuniversal in $t \in B$ for holomorphic deformations, then it there also is semiuniversal for real analytic deformations.*

In order to prove this proposition we extend a given real analytic deformation to a holomorphic deformation (cf. [DG]) and then use the semiuniversality for holomorphic deformations.

3 Construction of a Deformation with Special Properties

Proposition 3.1 *Let Y be a compact Riemann surface of genus p with m distinguished points $P_1, \dots, P_m \in Y$. Then there exists a real analytic deformation $\rho : Z \to V \subset \mathbb{C}^n$ with m distinguished cross sections $s_1, \dots, s_m : V \to Z$ such that $Y = Z_0$, $P_i = s_i(0)$, $\rho : Z \to V$ has bijective infinitesimal deformation in the zero point and the family $(Z' = Z \setminus \{s_1(V), \dots, s_m(V)\}) \to V$ is equal to $(\hat{Z}_0'/\Gamma_t)_{t \in V}$. Here $\hat{Z}_0$ donotes the universal covering of Z_0' and the Γ_t are defined as follows: There exist finitely many automorphisms of $\hat{Z}_0$ with coefficients depending real analytically on $t \in V$, which, for fixed $t \in B$, generate a subgroup Γ_t of the automorphism group of $\hat{Z}_0$ which acts properly discontinuously on $\hat{Z}_0$.*

We put $Y' = Y \setminus \{P_1, \dots, P_m\}$.
Let us first treat the case of hyperbolic fibers, i.e. the case $2p + m - 3 \geq 0$. Then the universal covering of Y' is the upper half plane H, and Y' can be represented as the quotient of H by a Fuchsian group Γ which is finitely generated and which acts fixpointfree and properly discontinuous on H. The set of generators of Γ can be assumed to be

$$\alpha_\nu = \frac{a_\nu z + b_\nu}{c_\nu z + d_\nu},\ \nu = 1, \dots, 2p, \quad \beta_\mu = \frac{\tilde{a}_\mu z + \tilde{b}_\mu}{\tilde{c}_\mu z + \tilde{d}_\mu},\ \mu = 1, \dots, m$$

where the determinants always are one, all coefficients are real and the α_ν are hyperbolic, the β_μ are parabolic ([A,p.42]). Sincs a relation has to be satisfied and H has a three parameter group we have $2(3p - 3 + m)$ real parameters from which all the coefficients can be recovered ([A,p.64]). Now we can construct the desired real analytic deformation: We take $V \subset \mathbb{C}^{3p-3+m}$ with $0 \in G$ and define $2p + m$ automorphisms

$$\frac{a_\nu(t)z + b_\nu(t)}{c_\nu(t)z + d_\nu(t)},\ \nu = 1, \dots, 2p, \quad \frac{\tilde{a}_\mu(t)z + \tilde{b}_\mu(t)}{\tilde{c}_\mu(t)z + \tilde{d}_\mu(t)},\ \mu = 1, \dots, m \tag{1}$$

of H with coefficients depending real analytically on t s.th. the coefficients satisfy our conditions (hence we only may define $2(3p - 3 + m)$ coefficient functions, the others are fixed by these conditions), the quotient of H by these automorphisms for $t = 0$ yields Y', and s.th. for no $\xi \in T_{V,0}$, $\xi \neq 0$ the derivatives of the coefficient functions in ξ–direction vanish simultaneously. Let Γ_t be the group generated by the automorphisms of (1) for $t \in V$. The proof of Theorem 1.2 can now be completed by showing:
1) If V is chosen small enough then for every $t \in V$ the group Γ_t acts properly discontinuously on H.
2) We can 'fill in the punctures' in the family $(H/\Gamma_t)_{t \in V}$ so that we get a real analytic deformation (of our kind).
3) This family has bijective infinitesimal deformation in the zero point.
Assertion 3) can be shown as follows: The restriction of our deformation to any real line through the zero point is, by construction, non stable in the zero point, hence, by Propo-

sition 2.2, the infinitesimal deformation of the total deformation is injective. Since, by the theorem of Riemann-Roch, we have $dim_{\mathbb{C}} H^1(Z_0, \Theta_0^*) = 3p - 3 + m = dim_{\mathbb{C}} V$, it is bijective. Proofs of 1) and 2) can be found in [DG].
We still have to deal with the case where $2p + m - 3 < 0$. If $p = 0$, $m \leq 2$ we have $dim_{\mathbb{C}} H^1(Z_0, \Theta_0^*) = 0$ and hence there is nothing to show. Only the case $p = 1$, $m = 0$, in which we have $dim_{\mathbb{C}} H^1(Z_0, \Theta_0^*) = 1$, remains. In this case Y can be represented as the quotient of the complex plane $\mathbb{C}$ by a group Γ generated by the two automorphisms $z \to z + 1$, $z \to ai + b$, $a, b \in \mathbb{R}$, $a > 0$ of $\mathbb{C}$. In the same way as in the hyperbolic case we now can define a real analytic deformation $(\mathbb{C}/\Gamma_t)_{t \in V} \to V$ of Y over a neighborhood $V = V(0) \subset \mathbb{C}$ by defining $\Gamma_t := < z \to z + 1, z \to z + ai + b + t >$. In the same way as in the proof of assertion 3) in the hyperbolic case it can be seen that the infinitesimal deformation in the zero point is bijective.

Now the proof of Theorem 1.2 is immediate: From Proposition 2.4 and Proposition 2.5 we conclude that the special real analytic deformations constructed in Proposition 3.1 are actually semiuniversal. Hence any real analytic deformation can locally be obtained by lifting from such a special real analytic family (cf. Definition 2.3), which completes the proof.

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