Deformation of Compact Riemann Surfaces Y of Genus p with Distinguished Points $P_1, \ldots, P_m \in Y$

Dedicated to E. Vesentini to the occasion of his 60^{th} birthday

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1 Introduction

In [G] an idea is given how in very general algebraic spaces X a jetmetric Λ can be constructed which has the following property:
If $C \subset X$ is a local complex curve then $\Lambda|_C$ is a hermitean metric with negative Gaussian curvature K, where K is bounded away from zero independantly of C and X is complete in resp. of this metric. If such a metric exists then X is hyperbolic in the sense of Kobayashi, c.f. [Kob]

The construction in [G] uses families of osculating algebraic curves along C which may be punctured if X is not compact. In this paper we show more generally for families of punctured Riemann surfaces:
The corresponding family of the universal coverings is real bianalytically and on the fibers holomorphically a cartesian product. Especially the family of the hyperbolic metrics on the punctured Riemann surfaces is a real analytic family

These results are obtained by proving that the so called Fricke coordinates of a (at least real analytic) family of punctured Riemann surfaces (see [Abi]) depend real analytically on the fiber parameter. This result probably was more or less known before by completely different methods, using quasi conformal mappings and the Teichmüller theory (c.f. [Abi], he uses the result that the Fricke coordinates are real analytic coordinates for the Teichmüller space without proof. From this result our results can be obtained quite easily by the theory of Teichmüller spaces). Our proof uses the general theory of deformation of complex spaces in higher dimensions. So it is shown that this meanwhile established theory also applies to the special case of deformation of punctured Riemann surfaces.

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2 Deformation of Punctured Compact Riemann Surfaces and Infinitesimal Deformation

Assume that X, B are real analytic manifolds and that $\pi : X \to B$ is a surjective, proper, smooth real analytic map.
Further assume that there exists a special atlas A of charts of X of the following form:

$$(w_1, w_2, z_1, \dots, z_{2n}) : U\,(\subset X) \to \mathbb{R}^{2n+2}$$

s.th. for every $x \in U$ $(z_1, \dots z_{2n})$ are real analytic coordinates around $\pi(x) \in B$; and s.th. for two such charts the w–components depend holomorphically on each other for fixed z–components.
The atlas A makes every fiber $X_t := \pi^{-1}(t)$, $t \in B$ a compact complex submanifold of X. We assume that every X_t is a Riemann surface of genus p. Moreover we assume that there are given real analytic cross sections $s_1(t), \dots, s_m(t)$ in X over B s.th. for every $t \in B$ the points $s_1(t), \dots, s_m(t)$ are disjoint.

Definition 2.1 *X is called a real analytic deformation of Riemann surfaces of genus p with m distinguished points. If additionally X, B are complex manifolds, A is a holomorphic atlas, and $\pi : X \to B$ and $s_1, \dots, s_m$ are holomorphic, then X is called a holomorphic deformation of Riemann surfaces of genus p with m distinguished points.*

In all what follows we mean by real analytic resp. holomorphic deformation these special kinds of deformations.

Next we define the infinitesimal deformations of such deformations in a point $t \in B$. We first assume that the deformation is real analytic. Let $\mathcal{E}_t$ be the tangent bundle defined by the charts of A, restricted to X_t, and $\mathcal{F}_t$ be the subbundle of those tangent vectors along the fibers. Let Θ_t resp. Ψ_t be the sheaves of holomorphic cross sections in $\mathcal{F}_t$ resp. $\mathcal{E}_t$. Further let Θ_t^* be the subsheaf generated by those sections of Θ_t which are zero in $s_1(t), \dots, s_m(t)$.
We take a covering $\mathcal{U} = \{U_1, \dots, U_l\}$ of open subsets $U_1, \dots, U_l$ in X_t s.th. for every $\xi \in T_{B,t}$, there exist fields $\eta_\nu \in H^0(U_\nu, \Psi_t|_{U_\nu})$ with
1) $\pi_*(\eta_\nu) \equiv \xi$
2) η_ν is tangent along the $s_1(t), \dots, s_m(t)$
Then the η_ν, $\nu = 1, \dots, l$ define a 0–cochain $\{\eta_\nu\} \in C^0(\mathcal{U}, \Psi_t)$ in the Čech complex defined by the sections of the sheaf Ψ_t and the covering $\mathcal{U}$. It yields a cocycle $d\{\eta_\nu\} \in Z^1(\mathcal{U}, \Theta_t^*)$ which, by passing to the limit, yields a cohomology class $\rho_t(\xi) \in H^1(X_t, \Theta_t^*)$. This is uniquely determined and $\rho_t : T_{B,t} \to H^1(X_t, \Theta_t^*)$ is a real linear map.
If the deformation is holomorphic then we define completely analogous using the holomorphic instead of the mixed real analytic and holomorphic structure. We get a complex linear map $\rho_t : T_{B,t} \to H^1(X_t, \Theta_t^*)$.

Definition 2.2 *The map $\rho_t : T_{B,t} \to H^1(X_t, \Theta_t^*)$ is called the infinitesimal deformation of the real analytic resp. holomorphic deformation.*

Definition 2.3 *A real analytic resp. holomorphic deformation is called stable in* $t \in B$ *(with resp. to* $s_1, \dots, s_m$ *) if it is a cartesian product over the double point in* t, *i.e. if there are finitely many charts* $(w^{(i)}, z) : U_i \to \mathbb{R}^{2n+2}$, $i = 1, \dots, l$ *out of the atlas* $\mathcal{A}$ *with:*

1) $X_t \subset \bigcup_{i=1}^{l} U_i$

2) $\frac{\partial s_j}{\partial z_k}(t) = 0$ *for* $j = 1, \dots, m$, z_k *being any component of* z

3) For any coordinate transformation $w^{(j)} = w^{(j)}(w^{(i)}, z)$ *we have* $\frac{\partial w^{(j)}}{\partial z_k}(t) = 0$.

We have the following

Theorem 2.4 *A real analytic resp. holomorphic deformation is stable in* $t \in B$ *with resp. to* $s_1, \dots, s_m$ *iff* $\rho_t(\xi) \equiv 0$ *in* $T_{B,t}$.

To prove that, it is first shown by using the long exact cohomology sequence belonging to the short exact sheaf sequence

$$0 \to \Theta_t^* \to \Psi_t \to \Psi_t / \Theta_t^* \to 0$$

that $\rho_t(\xi) \equiv 0$ iff for all $\xi \in T_{B,t}$ there exists a field $\eta \in \Psi_t(X_t)$ with $\pi_*(\eta) \equiv \xi$. Now it can easily be shown that the existence of such fields is equivalent with the existence of special charts in $\mathcal{A}$ like in Definition 2.3. For more details c.f. [KS,I] □

3 Existence of Semiuniversal Deformation

Definition 3.1 *A real analytic (resp. holomorphic) deformation* $\pi : X \to B$ *is called semiuniversal in* $t \in B$ *if:*

1) If $\psi : Z \to G$ *is an other real analytic (resp. holomorphic) deformation with* $0 \in G$ *and* $Z_0 = X_t$ *then there is a real analytic (resp. holomorphic) map* $\alpha : U(0) \to B$ *with* $\alpha(0) = t$ *s.th.* $X \circ \alpha = X \times_\alpha U$ *is isomorphic to* $Z|_U$ *under an isomorphism which is on* $Z_0 = X_t$ *the identity.*

2) The total derivative $d\alpha : T_{G,0} \to T_{B,t}$ *is uniquely determined.*

The semiuniversal deformation is uniquely determined always.
Remark: It is possible to prove that it is universal, i.e. that α itself is uniquely determined.

Lemma 3.2 *If* Y *is a compact Riemann surface of genus* p *with distinguished points* $P_1, \dots, P_m \in Y$ *there exists a holomorphic semiuniversal deformation* $\pi : X \to B \subset \mathbb{C}^n$ *with* $Y = X_0$, $P_i = s_i(0), i = 1, \dots, m$ *and* $n = \dim_{\mathbb{C}} H^1(X_0, \Theta_0^*)$

Proof. We have $H^2(Y, \Theta_t) = 0$. So if we forget about the distinguished points in Y for a moment there exists a seminuiversal deformation of compact Riemann surfaces (in the ordinary sense) $\pi' : X' \to B'$ with $X'_0 = Y$ and bijective infinitesimal deformation $\rho_0 : T_{B',0} \to H^1(X'_0, \Theta_0)$, cf. [KNS], [KS,II]

If we now additionally have holomorphic cross sections $s_1, \ldots, s_m$ through $P_1, \ldots, P_m$, we get $r := \max\{0, m - \max\{3-2p, 0\}\}$ additional deformation parameters, which are independant from the others So we have a holomorphic deformation (again of our kind, cf. Definition 2 1) $\pi : X \to B' \times \mathbb{C}^r$ s.th. $X_0 = Y$ and $\rho_0 : T_{B' \times \mathbb{C}^r, 0} \to H^1(X_0, \Theta_0^*)$ again is bijective The last two statements can be proved by applying the theorem of Riemann–Roch. Now we still have to show that this family is complete in $t = 0$ since then, by the bijectivity of ρ_0, it is semiuniversal in the zero point This can be proved like in [KS,II], but we also can get this immediately by using a general theorem stated in [Fl]. □

Lemma 3.3 *If $\pi : X \to B$ is semiuniversal in $t \in B$ for holomorphic deformations, then it there also is semiuniversal for real analytic deformations.*

Proof. Assume that $\Psi : Z \to G$ is a real analytic deformation with $0 \in G$ and $Z_0 = X_t$. We may assume that $G \subset \mathbb{R}^l$ with coordinates $x_1, \ldots, x_l$. We take a small domain $\tilde{G} \subset \mathbb{C}^l$ with complex coordinates $\tilde{x}_1, \ldots, \tilde{x}_l$ s.th. $G = \tilde{G} \cap \mathbb{R}^l$ and everything can be extended holomorphically onto $\tilde{G}$: Thus we get a holomorphic deformation $\tilde{\Psi} : \tilde{Z} \to \tilde{G}$ and disjoint cross sections $\tilde{s}_1, \ldots, \tilde{s}_m$. Since $\pi : X \to B$ is semiuniversal for holomorphic families we obtain a holomorphic map $\tilde{\alpha} : U(0) \to B$ with $\tilde{\alpha}(0) = t$ s.th. $\tilde{Z}|_U = X \circ \tilde{\alpha}$ and $d\tilde{\alpha}$ is uniquely determined.
By restricting $\alpha = \tilde{\alpha}|_{U \cap G}$ we have $Z|_{U \cap G} = X \circ \alpha$. Since $d\tilde{\alpha}$ maps infinitesimal deformation onto the same infinitesimal deformation and this remains true under restriction of $T_{\tilde{G},0}$ to $T_{G,0}$ and $\tilde{\alpha}$ to α the map $d\alpha$ is uniquely determined. □

4 Real Analytic Triviality of the Family of the Universal Coverings

We now wish to construct a real analytic deformation $\Psi : Z \to G$ with $Z_0 = Y$, where Y is again a given compact Riemann surface with distinguished points $P_1, \ldots, P_m$, and with bijective $\rho_0 : T_{G,0} \to H^1(Z_0, \Theta_0^*)$ and on G real analytic Fricke coordinates (c.f. [Abi] or see below). We need some preparations:
We put $Y' = Y \setminus \{P_1, \ldots, P_m\}$ and assume from now on $p \geq 2$ or $p = 1$ and $m \geq 1$ or $p = 0$ and $m \geq 3$. Then the universal covering of Y' is the upper half plane H, and Y' can be represented as the quotient of H by a Fuchsian group Γ which is finitely generated and which acts fixpointfree and properly discontinuous on H Let Π_1 be the fundamental group of Y'. It is generated by loops $\alpha_1, \alpha_1^*, \ldots, \alpha_p, \alpha_p^*, \beta_1, \ldots, \beta_m$ with the following properties:
1) The intersection number of α_ν, α_ν^* is 1
2) All the other loops only have the base point in common. Their intersection number is zero.
If $m = 0$ then the loops $\alpha_1, \alpha_1^*, \ldots, \alpha_p, \alpha_p^*$ generate Π_1 There is exactly one relation

$$\alpha_1 \cdot \alpha_1^* \cdot \alpha_1^{-1} \cdot (\alpha_1^*)^{-1} \cdot \ldots \cdot \alpha_p \cdot \alpha_p^* \cdot \alpha_p^{-1} \cdot (\alpha_p^*)^{-1} = 1.$$

If $m \neq 0$ Π_1 is generated by $\alpha_1, \ldots, \beta_m$ and is free with the $2p+m-1$ generators $\alpha_1, \ldots, \beta_{m-1}$. We have an isomorphism $\chi : \Pi_1 \to \Gamma$ Let

$$\chi(\alpha_\nu) = \frac{a_\nu z + b_\nu}{c_\nu z + d_\nu} \quad \chi(\alpha_\nu^*) = \frac{a_\nu^* z + b_\nu^*}{c_\nu^* z + d_\nu^*} \quad \chi(\beta_\nu) = \frac{\tilde{a_\nu} z + \tilde{b_\nu}}{\tilde{c_\nu} z + \tilde{d_\nu}}$$

where the determinants always are one and all coefficients are real. Since the $\chi(\alpha_\nu), \chi(\alpha_\nu^*)$ are hyperbolic, the $\chi(\beta_\nu)$ are parabolic ([Abi,p.42]), the relation has to be satisfied and H has a three parameter group we have $2(3p-3+m)$ real parameters from which all the coefficients can be recovered ([Abi,p.64]) (Both references don't depend on the Teichmüller theory or quasi conformal mappings.)
Now we can construct the desired real analytic deformation:
We take $G \subset \mathbb{C}^{3p-3+m}$ with $0 \in G$ and define $2p+m$ automorphisms

$$\frac{a_\nu(t)z + b_\nu(t)}{c_\nu(t)z + d_\nu(t)} \quad \frac{a_\nu^*(t)z + b_\nu^*(t)}{c_\nu^*(t)z + d_\nu^*(t)} \quad \frac{\tilde{a_\nu}(t)z + \tilde{b_\nu}(t)}{\tilde{c_\nu}(t)z + \tilde{d_\nu}(t)} \tag{1}$$

of H with coefficients depending real analytically on t s.th. the coefficients satisfy our conditions (hence we only may define $2(3p-3+m)$ coefficient functions, the others are fixed by these conditions), the quotient of H by these automorphisms for $t=0$ yields Y', and s.th. for no $\xi \in T_{G,0}$, $\xi \neq 0$ the derivatives of the coefficient functions in ξ-direction vanish simultaneously. Let Γ_t be the group generated by the automorphisms of (1) for $t \in G$ and $\Gamma_G := \Gamma_t, t \in G$. Then (possibly after having made G smaller) for all $t \in G$ the group Γ_t is properly discontinuous, fixpointfree and finitely generated:
Let D_0 be a fixed fundamental region of Γ_0. If G is small enough one can (possibly after having changed the generators of Γ_G for a moment), by the action of 'one letter words' $w(t) \in \Gamma_G$ in a small neighborhood of D_0, find 'fundamental regions' D_t which depend real analytically on t. Then we always have $w(t)(\dot{D}_t) \cap \dot{D}_t = \emptyset$ (where $\dot{D}_t$ denotes the open kernel of D_t), for longer words $g(t) \in \Gamma_G$, of course, that equation needn't hold. H is covered by the translates of D_t given by the elements of Γ_t for every $t \in G$. We have to show that the open kernels of those translates are disjoint.
Assume that for some t_0 two of them aren't disjoint. Then there also exists a $g_0(t_0) \in \Gamma_{t_0}$ with $\dot{D}_{t_0} \cap g_0(t_0)(\dot{D}_{t_0}) \neq \emptyset$. Let $\Phi : [0,1] \to G$ with $\Phi(0) = t_0$, $\Phi(1) = 0$ be any curve. If $g_0(0)(\dot{D}_0) \cap \dot{D}_0 = \emptyset$ we obtain by multiplying $m \geq 1$ 'letters' w_j from the left:
There exist open intervals $I_0, \ldots, I_l \subset \mathbb{R}$ with $\bigcup_{i=0}^{l} I_i \supset [0,1]$, $0 \in I_0$, $1 \in I_l$, $[0,1] \cap I_i \neq \emptyset$ for all $i = 0, \ldots, l$ and $g_i(t) := w_i(t) \cdot \ldots \cdot w_1(t)\ g_0(t) \in \Gamma_G$, $i = 1, \ldots, l$ with $g_i(t)(\dot{D}_t) \cap \dot{D}_t \neq \emptyset$ for $t \in \Phi(I_i \cap [0,1])$. We have $g_l(0) = id$ since Γ_0 was properly discontinuous and fixpointfree. Since the relations are kept while deforming Γ_0 to Γ_t we have $g_l = id$ on G. But then we have $g_{l-1}(t)(\dot{D}_t) \cap \dot{D}_t = w_l^{-1}(t)(\dot{D}_t) \cap \dot{D}_t = \emptyset$ by construction of D_t for all $t \in G$, which is a contradiction. So we have $g_0(0)(\dot{D}_0) \cap \dot{D}_0 \neq \emptyset$, hence $g_0(0) = id$ and then by the same argument as above we have $g_0 = id$ on G which proves our assertion.
Therefore $(H \times G)/\Gamma_G$ is a well defined family of Riemann surfaces with $H/\Gamma_0 = Y'$ on which $H \times G$ canonically yields real analytic and on the fibers holomorphic charts.

In order to get a real analytic deformation $\Psi : Z \to G$ like in Definition 2.1 from $(H \times G)/\Gamma_G$, we have to 'fill in the punctures'. If S resp. S' denote the underlying topological spaces of Y resp. of $Y' = Y \setminus \{P_1, \dots, P_m\}$ and $s_i(t) := (P_i, t) \in S \times G$, $i = 1, \dots, m$, that means that we can find real analytic and on the fibers holomorphic charts (with resp. to those charts on $(H \times G)/\Gamma_G$, the underlying topological space of which is by construction $S' \times G$ since the real analytic and on the fibers complex structure of $(H \times G)/\Gamma_G$ can be given through the automorphism $\chi_t : \Pi_1(S') \to \Gamma_t$) around all points $s_i(t) \in S \times G$, $t \in G, i = 1, \dots, m$, with the following properties:
All those charts give an atlas $\mathcal{A}$ (cf. Definition 2.1) on $S \times G$ in such a way that $(H \times G)/\Gamma_G = Z \setminus \{s_1(G), \dots, s_m(G)\}$, where Z denotes $S \times G$ with the structure given by $\mathcal{A}$, the fibers of Z are compact Riemann surfaces of genus p, and the $s_i(t)$, $t \in G$ are real analytic cross sections in Z. Let $Q = s_i(t_0)$ be arbitrary. Let V be a small neighborhood of Q in $S \times G$. Then by the isomorphism $\chi_t : \Pi_1(S') \to \Gamma_t$ the deck transformation $\beta_i(t)$ covers small simple loops in $S' \times \{t\}$ around the 'puncture' $s_i(t)$. $\beta_i(t)$ has exactly one fixed point $F(t) \in \partial\Delta$ (where we identify H and Δ through a fixed biholomorphic map) which depends real analytically on $t \in G$. So there exists a family of rotations in the fibers depending real analytically on t which maps $H \times G$ real bianalytically and on the fibers holomorphically onto itself s.th. the border points $F(t)$ are mapped to infinity. Now $\beta_i(t)$ is a translation $z \to z + b(t)$, where $b(t)$ depends real analytically on t. So the automorphism $(w, z) \to (b(t)^{-1}w, z)$ on $H \times G$ again is real bianalytic and on the fibers holomorphic. At last we map $H \times G$ to $\Delta \times G$ by $(w, z) \to (e^{iw}, z)$ Then the invers image of $V(Q)$ in $\Delta \times G$ is the intersection of an open neighborhood $W((0, t_0)) \subset \Delta \times G$ with $(\Delta \setminus \{0\}) \times G$ which, by construction and the properties of $\beta_i(t)$, is mapped real bianalytically and on the fibers holomorphically to $V \setminus s_i(G)$.
So we have constructed the desired charts and hence the real analytic deformation, the Fricke coordinates of which, i.e. the $2(3p - 3 + m)$ coefficient functions in (1) which weren't given by the restructions, depend by construction real analytically on G.
Since by the theorem of Riemann–Roch we have $\dim_{\mathbb{C}} H^1(Z_0, \Theta_0^*) = 3p - 3 + m$, we only have to show that the map $\rho_0 : T_{G,0} \to H^1(Z_0, \Theta_0^*)$ is injective.
Let $\xi = \sum_{\nu=1}^{2(3p-3+m)} a_\nu \frac{\partial}{\partial t_\nu} \in T_{G,0}$, $\xi \neq 0$, $L = \{t = (a_1, \dots, a_{2(3p-3+m)})s,\ s \in \mathbb{R}\} \cap G$ be the real line in ξ–direction, and 0_1 be the double point in $0 \in L$. If we restrict the transformations (1) to 0_1 their deformation is by construction not zero. Hence the restriction of $Z|_L$ to 0_1 has nonvanishing infinitesimal deformation: The complex structure on the fibers is parametrized by the elements of Γ_G up to the automorphism group of H. Since in the family (1) of generators of Γ_G the influence of this automorphism group already has been thrown out by a reduction of the number of free parameters (by our conditions for the coefficient functions) this means that the complex structures of the fibers are 'changing over 0_1'. Hence $Z|_L$ cannot be stable over 0_1, since otherwise there would exist real analytic and on the fibers biholomorphic coordinates which would make $Z|_L$ a cartesian product over 0_1, but then the complex structures of the fibers couldn't 'change over 0_1'. Hence the infinitesimal deformation of $Z|_L$ in the zero point isn't zero.
Since $\xi \in T_{G,0}, \xi \neq 0$ was arbitrary this means that the infinitesimal deformation of

$\Psi : Z \to G$ is injective in the zero point So our deformation $\Psi : Z \to G$ has all desired properties.

We now can prove the following:

Theorem 4.1 *Let $\pi' : X' \to B'$ be a real analytic deformation (like in Definition 2.1). Then its Fricke coordinates (more exactly those of $\pi' : X' \setminus \{s'_1, \ldots, s'_m\} \to B'$) depend real analytically on B'. Especially the universal covering $\hat{X}'$ of $X' \setminus \{s'_1, \ldots, s'_m\}$ is real bianalytically and on the fibers holomorphically equivalent to $H \times B'$, where H denotes the upper half plane, and the hyperbolic metric on the fibers of $X' \setminus \{s'_1, \ldots, s'_m\}$ depends real analytically on the fiber parameter $t \in B'$*

Proof: Let $t \in B'$ be arbitrary. Then with Lemma 3.2 there exists a holomorphic semiuniversal deformation $\pi : X \to B \subset \mathbb{C}^n$ with $X_0 = X'_t$ and $n = \dim_{\mathbb{C}} H^1(X_0, \Theta_0^*)$ From Lemma 3.3 we know that $\pi : X \to B$ is also semiuniversal for real analytic deformations. Together with our preceding construction this means:
1) There exists a real analytic mapping $\alpha' : U(t)(\subset B') \to B$ s.th. $X'|_U \cong X \circ \alpha'$
2) There exists a real analytic deformation $\Psi : Z \to G$ like constructed above with $Z_0 = X_0$ and with an in the zero point bijective infinitesimal deformation. There further exists a a real analytic map $\alpha : V(0) \to B$ with $Z|_V \cong X \circ \alpha$.
Since $d\alpha$ maps infinitesimal deformation to the same infinitesimal deformation and the infinitesimal deformation of $\Psi : Z \to G$ in 0 is bijective, $d\alpha$ is injective. Since $\dim_{\mathbb{C}} B = \dim_{\mathbb{C}} H^1(X_0, \Theta_0^*) = \dim_{\mathbb{C}} G$ the map $d\alpha$ is bijective. By the invers function theorem $\alpha^{-1} : B \to G$ exists locally around $0 \in B$. So we have $X \cong Z$ locally around 0 through the real bianalytic map α. Especially the Fricke coordinates of X depend real analytically on B since the Fricke coordinates of Z depend real analytically on G, both around the zero point. Since by 1) the family $\pi' : X' \to B'$ can be obtained by lifting from $\pi : X \to B$ the Fricke coordinates of X' depend real analytically on B'.
The other both assertions are an immediate consequence of this fact. □

Remark: In the cases of deformations with $p = 0, m \leq 2$ and $p = 1, m = 0$ (c.f. Definition 2.1) we cannot deform by using a Fuchsian group. In the case $p = 0, m \leq 2$ the deformation is trivial, and for $p = 1, m = 0$ the deformation is parametrized by the upper half plane H.

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