On the Infinitesimal Deformation of Simply Connected Domains in One Complex Variable

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1 Introduction

1. We denote by D a bounded connected domain with $0 \in D$ and by Δ the open unit disk, both in the complex plane $\mathbb{C}$ with the variable z, and by X a bounded complex domain in $\mathbb{C} \times D$ with the following properties:

1) The projection $\pi : (w,z) \to z$ maps X surjectively on D. X_z denotes the fiber $\pi^{-1}(z) \cap X = (\mathbb{C} \times \{z\}) \cap X$ for $z \in D$. All X_z are homeomorphic to Δ.

2) For all $z \in D$ there is a neighborhood $V(z)$ and a neighborhood $U(\partial X) \subset \mathbb{C} \times V(z)$, where ∂X denotes the intersection of the boundary of X with $\mathbb{C} \times D$, s.th. by a real analytic and in w holomorphic coordinate transformation

$$\tilde{w} = f(w,z)\,,\ \tilde{z} = z$$

in U with coordinates (w,z) the set $\partial X \cap U$ is in the new coordinates $(\tilde{w}, \tilde{z})$ the cartesian product $\partial X_0 \times V(z)$.

3) ∂X_0 is picewise smooth: That means here:
There is a neighborhood $V(\partial X_0) \subset \pi^{-1}(0)$ in which there are finitely many smooth one dimensional real analytic subsets $A_1, A_2, ..., A_n$ s.th. $\partial X_0 \subset \bigcup_{i=1}^{n} A_i$. In one point of ∂X_0 there intersect at most two different A_ν, A_μ and the intersection is transversal there.

It can be proved very easily that after having made V sufficiently small, there are only finitely many intersection points of this kind. If in addition $f(w,z)$ is holomorphic in (w,z), these assumptions are very restrictive, of course. It follows from them that ∂X is an analytic hypersurface in the sense of Behnke–Thullen [BT,p.30]. Namely ∂X has a fibration in one codimensional analytic sets.
To simplify writing, we introduce the following convention:
If we say that we do something *z–locally*, we mean that we do this in $\mathbb{C} \times V(z)$, where $V(z)$ is a small neighborhood of z in D (e.g. if we say that there z–locally exists a map $F : X \to X'$ onto a fibration X' we mean that F and X' only need to be defined over such a neighborhood $V(z)$). We only write 'z–locally' in those steps of a proof in which we actually have to restrict onto such a neighborhood, in further steps it is no longer mentioned, but, of course, we stay over such a neighborhood for the rest of the proof if we only have restricted to it for one time.
In section 2 we shall prove that z-locally the fiber space X can be transformed by a biholomorphical fiber preserving coordinate transformation into the fiber space X^*, s.th. X^* has the properties 1) to 3) and ∂X_0^* is smooth. With 'fiber preserving', we mean from now on that the map is the identity in the second component.

2. The assumptions 1) to 3) of subsect. 1.1 are satisfied in the following case:
Denote by X a bounded complex domain in $\mathbb{C} \times D$ again, but now with these properties:

1) $\pi : (w,z) \to z$ maps X surjectively onto D and all X_z are homeomorphic to Δ.

2) The boundary ∂X is real analytically smooth.

3) The map $\pi : \partial X \to D$ is smooth.

Proof: We z-locally have to construct the coordinate transformation $\tilde{w} = f(w,z)$, $\tilde{z} = z$, to simplify writing we assume $z = 0$. We may assume that ∂X_0 is the boundary of the unit disk: We can obtain that by a coordinate transformation in w which is holomorphic in a neighborhood of $\overline{X_0}$ which can be constructed with the Schwarz reflection principle (Firstly there exists a biholomorphic mapping from X_0 to Δ by the Riemann mapping theorem, which can be extended into a neighborhood of $\overline{X_0}$ in $\mathbb{C} \times \{0\}$ by the Schwarz reflection principle and which is still homeomorphic in $\overline{X_0}$. Then it can easily be shown that this extension still is injective in such a neighborhood if it is sufficiently small (c.f. the proof of Lemma 2.1, where this is done in a more general situation)). 0–locally we can write

$$\partial X = \{(w,z) : \log|w| = h(\theta, z)\}$$

with $\theta = arc(w)$ and h being periodic in θ and real analytic in θ and z. The last property of h can be derived from assertion 3) with the implicit function theorem for real analytic functions [GR,p.105] after having introduced logarithmic coordinates in a neighborhood of ∂X_0 in $\mathbb{C} \times D$. 0–locally there now exists a neighborhood $U(\partial X)$ of ∂X and a holomorphic function $\tilde{f}(w,z)$ with $\tilde{f} = h$ on ∂X – this follows by a standart argument from the fact that ∂X is real analytically smooth. Now define

$$f(w,z) = e^{\log(w) - \tilde{f}(w,z)} \quad ; \quad \tilde{w} = f(w,z)\,,\ \tilde{z} = z$$

on $U(\partial X)$. If, if necessary, we make $U(\partial X)$ smaller, this 0–locally yields a coordinate transformation. In the new coordinates defined in $U(\partial X)$ each fiber of ∂X is the boundary of the unit disk. □

3. In the sects. 3 and 4 we shall assume that X is of the type of subsect. 1.1 and $f(w,z)$ is holomorphic or that X is of the type of subsect. 1.2. We moreover assume a holomorphic cross section s in X over D. In sect. 3 for each point $z \in D$ the square integral $r_s(z)$ of the infinitesimal deformation of $\overline{X_z}$ with respect to s is defined. Among others it is shown that $(\overline{X}, s)$ is stable in z iff $r_s(z) = 0$ and that for those X which fulfill both conditions mentioned above $(\overline{X}, s)$ is a cartesian product in a neighborhood of some $z_0 \in D$ iff $r_s(z) \equiv 0$ there.
By the hyperbolic metric on a simply connected domain in $\mathbb{C}$ we understand the (uniquely determined) complete real analytic hermitean metric with Gauss curvature -1. We now equip every fiber X_z with this hyperbolic metric and prove:
We obtain a real analytic family of hermitean metrics $ds^2 = g(w,z)dwd\overline{w}$ on the fibers X_z . If X is of the first type mentioned above we can, as an application of sect. 3, construct a hermitean metric on D from the hyperbolic metrics on the X_z which has non positive curvature which is zero iff $r_s(z) = 0$: We denote by $\chi(z)$ a holomorphic field on $s(z)$ of nonvanishing vectors in direction of the fibers. We obtain on D a hermitean metric

$$ds^2 = h(z)dzd\overline{z} = g(s(z), z)|\chi(z)|^2 dzd\overline{z}$$

It is shown for the Gauss curvature $\mathcal{K}(z)$:

$$\mathcal{K}(z) = -\epsilon(z) \cdot r_s(z)$$

where $\epsilon(z) > 0$ is a function which may depend on $\chi(z)$ and $s(z)$. As it is done in [G], this construction can be generalized to manifolds. Then it gives a principle to construct on a cross section in a smooth family of hyperbolic Riemann surfaces a metric with non positive curvature which is zero iff the square integral of the infinitesimal deformation vanishes.

2 The Riemann Mapping Theorem with Parameters

1. We prove the following extension lemma:

Lemma 2.1 *Let $\mathbb{C}$ have the variable $w = u + iv$ and the bounded subdomain $B \subset \mathbb{R}^n$ the variable $x = (x_1, ..., x_n)$. Let $\pi : (w, x) \to x$ be the canonical projection, it gives a fibration of $\mathbb{C} \times B$. Let G, H be bounded subdomains of $\mathbb{C} \times B$ and $G_x = \pi^{-1}(x) \cap G$, $H_x = \pi^{-1}(x) \cap H$ for all $x \in B$ be m–fold connected. Let G and H have real analytically smooth boundaries in $\mathbb{C} \times B$ for which the projections $\pi : \partial G \to B$ and $\pi : \partial H \to B$ are smooth. Let $F : G \to H$ be a bi real analytic fiber preserving map which additionally is holomorphic on the fibers. Then for every $x_0 \in B$ there exists a neighborhood $U(x_0) \subset B$, a neighborhood V of $\overline{G} \cap (\mathbb{C} \times U)$ in $\mathbb{C} \times U$ and a bi real analytic fiber preserving and in the fibers holomorphic extension*

$$\hat{F} : V \to \mathbb{C} \times B \text{ of } F.$$

If in addition n is even and F is holomorphic then $\hat{F}$ is holomorphic.

Proof: Since all G_x and H_x are m–fold connected $F : G_x \to H_x$ can be extended to a topological map $F_x : \overline{G_x} \to \overline{H_x}$ in a unique way (c.f. [BS,p.371]). Let w_0 be arbitrary with $(w_0, x_0) \in \partial G_{x_0}$. Then from the facts that ∂G and ∂H are real analytically smooth and π is smooth on ∂G and ∂H, one can derive by using the implicit function theorem for power series (c.f. [GR,p.105]) and then standart methods (like e.g. in the proof of [BS,p.373]): There exist neighborhoods $W((w_0, x_0))$ and $W'(F_{x_0}(w_0, x_0))$, open subsets O, O' in $\mathbb{C} \times \mathbb{R}^n$ and bi real analytic fiber preserving and on the fibers holomorphic mappings

$$K : W \to O;\ (w, x) \to (k(w), x) \quad K' : W' \to O';\ (w, x) \to (k'(w), x)$$

s.th. $K(W \cap G) = \{(w, x) \in O : \Im w > 0\}$, $K'(W' \cap H) = \{(w, x) \in O' : \Im w > 0\}$ and $F(W \cap G) \subset W' \cap H$.
Then $K' \circ F \circ K^{-1} : K(W \cap G) \to K'(W' \cap H)$ is bi real analytic, fiber preserving, in w holomorphic and can be extended fiberwise topologically into the borders. We can reflect this map for every fixed $x \in B$ at the real axix of O like in [BS,p.188] and obtain an extension $\tilde{F}$ into a neighborhood of $K((w_0, x_0))$ in O which is holomorphic in w for every fixed $x \in B$ (c.f. [BS,p.187f]) and is real analytic in every point of O which isn't lying on $O \cap \{\Im w = 0\}$. Now we can lift back the mapping $\tilde{F}$ and get an extension of F into a neighborhood of (w_0, x_0).
Since in this construction $(w_0, x_0) \in \partial G_{x_0}$ was arbitrary, we can cover ∂G_{x_0} with finitely many of those neighborhoods. If we had made them small enough before selecting the finite covering the extensions of F on each of such neighborhoods glue together to one extension $\hat{F}$, because on each fiber $\pi^{-1}(x)$ the extension of the holomorphic mapping F is uniquely determined by the identity lemma for holomorphic functions in a small neighborhood of $\overline{G_x}$.
Since we have covered ∂G_{x_0} by in $\mathbb{C} \times B$ open sets, we have:
There exists a neighborhood U of x_0 in B s.th. $\hat{F}$ is an extension of F into a neighborhood V of $\overline{G} \cap (\mathbb{C} \times U)$.
We still have to show that $\hat{F}$ is real analytic in the points of ∂G: For every $x_1 \in U$ we may assume that there exists a cycle $C \subset V_{x_1} \setminus \overline{G_{x_1}}$ consisting of m smooth closed curves s.th. $\overline{G_{x_1}}$ is in the interior of C and that this holds for all x in a sufficiently small neighborhood

$U'(x_1)$ in B. We make the Cauchy integral representation of $\hat{F}$ (strictly speaking of the first component of $\hat{F}$) with resp. to C. Since $\hat{F}$ is real analytic in $V \setminus \overline{G}$ this property especially holds on $C \times U'$ and so also in $\overline{G} \cap (\mathbb{C} \times U')$ since $\hat{F}$ can be expressed by $\hat{F}$ on $C \times U'$ by the Cauchy integral formula.
It remains to show that we can make U and V so small that $\hat{F}$ maps V bi real analytically onto an open subset of $\mathbb{C} \times U$. With [GR,p.92] it suffices to show that $\hat{F}$ is injective there. At the beginning of this proof we have already shown that $\hat{F}$ is injective on $\overline{G_{x_0}}$. Since $\hat{F}$ is holomorphic in a neighborhood of $\overline{G_{x_0}}$ in $\mathbb{C} \times \{x_0\}$ and F maps G_{x_0} biholomorphically onto H_{x_0} and both sets have smooth boundaries it follows that $\hat{F}_w \neq 0$ in every $(w_0, x_0) \in \partial G_{x_0}$, otherwise H_{x_0} would have an edge in $\hat{F}((w_0, x_0))$ or F couldn't be injective in G_{x_0}. Since $\hat{F}$ is fiber preserving we therefore have that its functional determinant doesn't vanish in any point of $\overline{G_{x_0}}$, hence it is locally bijective around each of those points in resp. of $\mathbb{C} \times B$. If there wouldn't exist small U and V defined like above s.th. $\hat{F}$ is globally injective there we would have two sequences $(P_\nu)_{\nu \in IN}$, $(Q_\nu)_{\nu \in IN}$ of points of V with $\mathrm{dist}(P_\nu, \overline{G_{x_0}}) < \frac{1}{\nu}$ (and the same for the other sequence) with the property $\hat{F}(P_\nu) = \hat{F}(Q_\nu)$. From this it follows that there exist points P, Q in $\overline{G_{x_0}}$ and subsequences of $(P_\nu)_{\nu \in IN}$ resp. $(Q_\nu)_{\nu \in IN}$ with $\hat{F}(P_\nu) = \hat{F}(Q_\nu)$ which converge to P resp. Q. Since $\hat{F}$ is injective on $\overline{G_{x_0}}$ we have $P = Q$, which is a contradiction, since $\hat{F}$ is locally injective around P. Since the fact that $\hat{F}$ is holomorphic if F was immediately follows from the identity lemma for power series applied to the $\overline{\partial}$–derivatives of $\hat{F}$ everything has been proved. □

For fibrations with edges we need another extension lemma:

Lemma 2.2 *Let X resp. X' be fibrations of the type described in subsect. 1.1 resp. in subsect. 1.2. Let $F : X \to X'$ be a fiber preserving, bi real analytic and on the fibers holomorphic map. Then F can be extended to a topological map $\tilde{F} : \overline{X} \to \overline{X'}$ which is real analytic in all points of ∂X except the edges.*

Proof: With [BS,p.364] we can extend F topologically into the border on each fiber. This gives us a bijective extension $\tilde{F} : \overline{X} \to \overline{X'}$ of F for which we want to prove that it is topological and real analytic in all border points (w_0, z_0) of X except the edges. To simplify writing we assume $z_0 = 0$. We start with the second assertion. Let $P = (w_0, 0) \in \partial X_0$ be no edge. According to property 2) of subsect. 1.1, we have 0–locally a real analytic and in the fibers holomorphic coordinate transformation in $U(\partial X)$ from coordinates (w, z) to coordinates $(\tilde{w}, z)$ s.th. in the latter we have $\partial X = \partial X_0 \times D$. In these coordinates let $P = (\tilde{w}_0, 0)$. From the Riemann mapping theorem we know that there exists a biholomorphic map $r : X_0 \to \Delta$. First this map can be extended topologically into the borders by [BS,p.364], then it can be holomorphically extended into a neighborhood $V(P)$ in $\mathbb{C} \times \{0\}$. With the same argument as in the proof of Lemma 2.1 we can show that V can be chosen so small that the extension $\tilde{r}$ of r maps $X_0 \cup V$ biholomorphically to $\Delta \cup \tilde{r}(V)$. Since in coordinates $(\tilde{w}, z)$ ∂X is a cartesian product we can extend $\tilde{r}$ canonically onto the other fibers, then it yields real analytic and on the fibers holomorphic coordinates $(w^*(\tilde{w}), z)$ in $(X \cap U) \cup W$, where W is a small neighborhood of P in U. This patches together $X \cup U$ and $(\mathbb{C} \times D) \setminus (X \setminus U)$ along $(X \cap U) \cup W$ and we obtain a complex manifold $\check{Y}$ over D. In the coordinates (w^*, z) in $(\mathbb{C} \times D) \setminus (X \setminus U)$, we have $\partial X = \partial\Delta \times D$, hence ∂X is real analytically smooth there. Now we can use Lemma 2.1 to this coordinate system and get a fiber preserving real analytic and on the fibers holomorphic extension of F from a neighborhood of $\overline{X}$ in $\check{Y}$ in coordinates (w^*, z) to a neighborhood of $\overline{X'}$ (the interior boundary can 0–locally be chosen as $\partial(\epsilon \cdot \Delta) \times D$ with some $\epsilon < 1$). If we draw this extension back through the both coordinate transformations we get a fiber preserving real analytic and on the fibers holomorphic extension of F into a neighborhood of P, which, since $\tilde{F}$ was topologically in every fiber, is equal to $\tilde{F}$ on the intersection of this neighborhood with ∂X. So the first

assertion has been proved.
To show that $\tilde{F}$ also is continuous in the edges of $\overline{X}$, we can work in $U(\partial X)$ in coordinates $(\tilde{w}, z)$, where ∂X is a cartesian product. Let $(\tilde{w}_0, 0)$ be an edge point. Let $V((\tilde{w}_0, 0)) \subset\subset (U \cap \overline{X})_0$ be a neighborhood with continuous boundary which is so small that its closure in $\overline{X_0}$ doesn't contain any other edge of X_0. Then 0–locally we have $\overline{V} \times D \subset U$, where the cartesian product is meant in $(\tilde{w}, z)$–coordinates, and $\tilde{F}((V \setminus \{(\tilde{w}_0, 0)\}) \times D)$ is a subset of $\overline{X'}$ which is cut off from this set by a continuous hypersurface. In every fiber of this subset exactly one borderpoint is missing to $\tilde{F}(V \times D)$, and this must be the image of the edge lying in $V \times D$. Now take a sequence $((\tilde{w}_n, z_n))_{n \in IN}$ converging against $(\tilde{w}_0, 0)$ which may be assumed lying in $V \times D$ with $\tilde{w}_n \neq \tilde{w}_0$ (use the continuity of $\tilde{F} : \overline{X_z} \to \overline{X'_z}$). Then every subsequence of $(\tilde{F}((\tilde{w}_n, z_n)))_{n \in IN}$ must have an accumulation point in $\tilde{F}(\overline{V} \times \{0\})$. If there exists one with an accumulation point different from $\tilde{F}((\tilde{w}_0, 0))$, choose a subsequence of $F((\tilde{w}_n, z_n)))_{n \in IN}$ converging against it. Then the invers image subsequence must converge against a point in $(\overline{V} \setminus \{(\tilde{w}_0, 0)\}) \times D$ by the continuity of $\tilde{F}^{-1}$ outside the edges, what is a contradiction, since the invers image subsequence converges against $(\tilde{w}_0, 0)$. So $\tilde{F}$ is continuous everywhere. This completes the proof of this Lemma. □

2. Assume that $Y \subset \mathbb{C}$ is a bounded domain with picewise smooth boundary ∂Y. We first solve the Dirichlet problem for the cartesian product $Y \times D$.

Lemma 2.3 *Assume that $p(w, z)$ is a complex real analytic function on the boundary $\partial Y \times D$. Then there is a unique continuous function $q(w, z)$ in $\overline{Y} \times D$ such that*

1. *$q|_{Y \times D}$ is real analytic*
2. *$q|_{\partial Y \times D} \equiv p$*
3. *$q|_{Y \times \{z\}}$ always is harmonic in w*

If moreover $C \subset Y \times D$ is a compact subset, we have with positive real numbers d, δ, ϵ which only depend on p and C:

$$\left|\frac{\partial^{\nu+\mu+\lambda}}{\partial w^{\nu} z^{\mu} \overline{z}^{\lambda}} q(w, z)\right| < d \cdot \epsilon^{\nu} \cdot \delta^{\mu+\lambda} \cdot \nu! \cdot \mu! \cdot \lambda! \tag{1}$$

The same inequalities hold for $\overline{w}$ instead of w.
If $p(w, z)$ only is defined z–locally, everything holds z–locally.

Proof: By the classical Dirichlet problem we can extend harmonically on each fiber. Therefore there exists a complex function q in $\overline{Y} \times D$ with the properties 2. and 3.. We have to prove that q is continuous in $\overline{Y} \times D$ and real analytic in $Y \times D$. The continuity follows by the maximum principle: If z_1, $z_2 \in D$ are very near then $\sup_{w \in \partial Y} |p(w, z_1) - p(w, z_2)|$ is small. The difference $q(w, z_1) - q(w, z_2)$ is harmonic in Y, hence it is also small in $\overline{Y}$. To get "real analytic", we need estimates for the derivatives for the continuing function in the classical Dirichlet problem. Assume that $K \subset Y$ is a compact subset. Then there exists a positive real number $\epsilon = \epsilon(K)$ s.th. for an in Y harmonic and in $\overline{Y}$ continuous function h and for every $w \in K$ we have the inequalities

$$\left|\frac{\partial^{\nu} h(w)}{\partial w^{\nu}}\right| < \nu! \cdot \epsilon^{\nu} \cdot \sup_{w \in \partial Y} |h(w)| \tag{2}$$

The same inequalities hold for $\overline{w}$. We can derive that for $\nu = 1$ with a central symmetric smoothing function, for $\nu > 1$ we additionally can use the Cauchy inequalities since h_w resp. $h_{\overline{w}}$ are holomorphic resp. antiholomorphic. Since p can be real analytically extended into a neighborhood of $\partial Y \times D$, we can derive from that by standart methods that for any $z_0 \in D$ there exists a neighborhood $U(z_0) \subset D$ and positive real numbers $\delta,\, d$ with

$$\sup_{z\in U,\, w\in \partial Y} |\frac{\partial^{\mu+\lambda} p(w,z)}{\partial z^{\mu} \overline{z}^{\lambda}}| < \mu! \cdot \lambda! \cdot d \cdot \delta^{\mu+\lambda} \tag{3}$$

If we apply on the harmonic continuation $q_{\mu,\lambda}(w,z)$ of $\frac{\partial^{\mu+\lambda} p}{\partial z^{\mu} \overline{z}^{\lambda}}$ (2) and then (3) we get

$$\sup_{z\in U,\, w\in K} |\frac{\partial^{\nu} g_{\mu,\lambda}(w,z)}{\partial w^{\nu}}| < \nu! \cdot \mu! \cdot \lambda! \cdot d \cdot \epsilon^{\nu} \cdot \delta^{\mu+\lambda} \tag{4}$$

the same inequalities hold for $\overline{w}$. So in a neighborhood of any (w_0, z_0) with $w_0 \in K$ the power series

$$\sum_{\nu,\mu,\lambda=0}^{\infty} \frac{1}{\nu!\mu!\lambda!} (\frac{\partial^{\nu}}{\partial w^{\nu}} q_{\mu,\lambda}(w_0, z_0))(w - w_0)^{\nu} (z - z_0)^{\mu} (\overline{z} - \overline{z_0})^{\lambda} +$$

$$+ \sum_{\nu,\mu,\lambda=0}^{\infty} \frac{1}{\nu!\mu!\lambda!} (\frac{\partial^{\nu}}{\partial \overline{w}^{\nu}} q_{\mu,\lambda}(w_0, z_0))(\overline{w} - \overline{w_0})^{\nu} (z - z_0)^{\mu} (\overline{z} - \overline{z_0})^{\lambda}$$

converges uniformly. It can now easily be shown that this power series is the function $q(w, z)$ which hence is real analytic in (w_0, z_0). (1) now follows from (4). $\square$

3. Let X be like in subsect. 1.1 and the holomorphic cross section s like in subsect. 1.3. We prove:

Lemma 2.4 *There z–locally exists a topological fiber preserving map* $F : \overline{X} \to \overline{\Delta} \times D$ *with the following properties:*

1) The cross section s is mapped onto $0 \times D$

2) Every fiber X_z is mapped biholomorphically onto $\Delta \times \{z\}$

3) The map $F|_X$ is bi real analytic.

Proof:
1) Firstly we do a fiber preserving biholomorphic transformation of $\mathbb{C} \times D$ s.th. s becomes the zero cross section $0 \times D$. The new coordinates of $\mathbb{C} \times D$ are again denoted by (w, z). The properties 1) to 3) of X listed in subsect. 1.1 are still satisfied.
To simplify writing we assume $z = 0$. Next we introduce some notations: There 0–locally is a domain $H \subset\subset U(\partial X)$ which is a cartesian product with respect to the coordinates $(\tilde{w}, \tilde{z})$ s.th. $\partial X_0 \subset H_0$, H_0 is homeomorphic to a ring of circles, ∂H_0 consists of two disjoint smooth real analytic curves and s.th. $\partial X_0 \times D$ has a positive distance from ∂H. Next we 0–locally can find a simply connected domain $G_0 \subset\subset X_0$ with smooth real analytic boundary s.th. $\partial G_0 \times D \subset\subset X \cap H$. We put $S' = \partial G_0 \times D$, $X' = G_0 \times D$, $X'' = X \cap H$ and denote by S'' the interior boundary of H. Then we have $X' \cup X'' = X$ and $S' \subset X''$ resp. $S'' \subset X'$ have positive distances from $\partial X''$ resp. from $\partial X'$.
2) The main step of the proof is to construct a continuous function $q : \overline{X} \to \mathbb{R}$ with $q|_{\partial X} \equiv -\log |w|$ which is real analytic in X and harmonic in every fiber of X.
This can be done by the following iteration:
In the n^{th} step $(n \geq 1)$, we first apply Lemma 2.3 on X'' in coordinates $(\tilde{w}, \tilde{z})$ with boundary values p''_n defined as 0 on ∂X and as $q'_{n-1} + \log |w|$ on S'' and get the continuation $q''_n : \overline{X''} \to \mathbb{R}$. Then we apply Lemma 2.3 on X' in coordinates (w, z) with boundary values $p'_n = q''_n - \log |w|$ on S' and get the continuation $q'_n : \overline{X'} \to \mathbb{R}$.

We put $q'_0 \equiv 0$ and get two sequences $(q''_n)_{n\in IN}$ resp. $(q'_n)_{n\in IN}$ defined on $\overline{X''}$ resp. on $\overline{X'}$. The boundary values of $q'_{n+1} - q'_n$ on S' are $q''_{n+1} - q''_n$, the boundary values of $q''_{n+1} - q''_n$ are $q'_n - q'_{n-1}$ on S'' and 0 on ∂X. Hence by the maximum principle we get:

$$\sup_{(w,z)\in S''} |q'_{n+1} - q'_n| \leq \sup_{(w,z)\in S'} |q''_{n+1} - q''_n|$$

$$\sup_{(w,z)\in S'} |q''_{n+1} - q''_n| \leq \epsilon \cdot \sup_{(w,z)\in S''} |q'_n - q'_{n-1}|$$

with a real number $\epsilon < 1$ which only depends on the choice of X' and X''. The reason for the existence of such an ϵ is that $q''_{n+1} - q''_n$ cannot take its maximum on S' because this function is harmonic on X'' and 0 on ∂X. It is easily shown that one gets the same ϵ for all in w harmonic functions on X'' which vanish on ∂X. From these inequalities it follows that the sequences $(q'_n)_{n\in IN}$ resp. $(q''_n)_{n\in IN}$ converge to limits q' resp. q'' on X' resp. X'' uniformly in w and locally uniformly in z, especially q' and q'' are harmonic on each fiber. We have $q' + \log|w| = q''$ on S' and on S'', hence this equation follows in $X' \cap X''$ and we can define:

$$q = \begin{cases} q' & on\ X' \\ q'' - \log|w| & on\ X'' \end{cases}$$

q is harmonic in every fiber of X and continuous in $\overline{X}$ with $q|_{\partial X} = -\log|w|$.

We still have to prove that q is real analytic in X. It suffices to prove that 0–locally we have $q|_{S'}(w,z) = \sum_{\nu\in IN^2} a_\nu(w) z^{\nu_1}\overline{z}^{\nu_2}$ with in S'_0 continuous functions $a_\nu(w)$, where the series converges absolutely uniformly in $w \in S'$, since then the $\frac{\partial^{\mu+\lambda}}{\partial z^\mu \overline{z}^\lambda}$–derivatives of $q|_{S'}$ exist and are continuous, and to prove that for them the inequalities (3) hold. Then we can prove exactly like in Lemma 2.3 that q is real analytic in X'. From this we have that $q|_{S''}$ is real analytic and an application of Lemma 2.3 yields that $q|_{X''}$ is real analytic.
Let $r(w,z)$ be real analytic in a neighborhood of S'_0 in U. Let
$r(w,z) = \sum_{\nu\in IN^2} a_\nu(w) z^{\nu_1}\overline{z}^{\nu_2} = \sum_{\nu\in IN^2} \tilde{a}_\nu(\tilde{w}) \tilde{z}^{\nu_1}\overline{\tilde{z}}^{\nu_2}$. For real $t > 0$, we put

$$\|r\|_{S',t} := \sum_{\nu\in IN^2} \sup_{(w,0)\in S'_0} |a_\nu(w)| t^{\nu_1+\nu_2}\ ,\ \|r\|^{\sim}_{S',t} := \sum_{\nu\in IN^2} \sup_{(\tilde{w},0)\in S'_0} |\tilde{a}_\nu(\tilde{w})| t^{\nu_1+\nu_2}$$

$\|r\|_{S'',t}$ and $\|r\|^{\sim}_{S'',t}$ are defined in the same way for a function r which is real analytic in a neighborhood of S''_0. $\|r\|_{S',t}$ resp. $\|r\|^{\sim}_{S'',t}$ are also well defined if r only is real analytic in a neighborhood of S'_0 in S' resp. in a neighborhood of S''_0 in S''. We further put

$$d_n := q''_{n+1} - q''_n : \overline{X''} \to I\!R\ ,\ e_n := q'_{n+1} - q'_n : \overline{X'} \to I\!R$$

and have: $d_n|_{\partial X} \equiv 0$, $d_n|_{S''} = e_{n-1}|_{S''}$ $e_n|_{S'} = d_n|_{S'}$.
3) The first step of the proof is to choose two positive real numbers s and t with special properties.
Let $h(\tilde{w})$ be continuous in $\overline{X''_0}$ with $h|_{\partial X_0} \equiv 0$ and harmonic in X''_0. Let $(\tilde{w}_0, 0) \in S'_0$ and $h(\tilde{w}) = \sum_{\mu\in Z\!\!Z} (\tilde{w} - \tilde{w}_0)^\mu \tilde{a}_\mu(\tilde{w}_0, 0)$ be the power series expansion (where we set $(\tilde{w} - \tilde{w}_0)^\mu = (\overline{\tilde{w}} - \overline{\tilde{w}_0})^{-\mu}$ for negative μ). From (2) we know

$$\sup_{(\tilde{w}_0,0)\in S'_0} |\tilde{a}_\mu(\tilde{w}_0, 0)| < \epsilon^{|\mu|} \sup_{(\tilde{w}_0,0)\in C} |h(\tilde{w}_0, 0)|$$

where C is a smooth curve lying in X_0 between S'_0 and S''_0, touching none of them, and ϵ only depends on C. Then with the same argument as in 2) we get a $\delta < 1$ depending only on C s.th.

$$\sup_{(\tilde{w}_0,0)\in C} |h(\tilde{w}_0, 0)| < \sup_{(\tilde{w}_0,0)\in S''_0} |h(\tilde{w}_0, 0)| \cdot \delta$$

Now we choose a real number $r > 0$ so small that $\rho := \delta(1+r)^2 < 1$ and then $t > 0$ so small that $\sum_{\mu\in\mathbb{Z}} \epsilon^{|\mu|}t^{|\mu|} < 1+r$, especially t is independant of h. Then we have got:

$$\sum_{\mu\in\mathbb{Z}} t^{|\mu|} \sup_{(\tilde{w}_0,0)\in S_0'} |\tilde{a}_\mu(\tilde{w}_0,0)| < \delta(1+r) \sup_{(\tilde{w}_0,0)\in S_0''} |h(\tilde{w}_0,0)| \tag{5}$$

If h is continuous in $\overline{X_0'}$ and harmonic in X_0', we get with the same methods: There exists a $t > 0$ independant of h s.th.

$$\sum_{\mu\in\mathbb{Z}} t^{|\mu|} \sup_{(w_0,0)\in S_0''} |a_\mu(w_0,0)| < (1+r) \sup_{(w_0,0)\in S_0'} |h(w_0,0)| \tag{6}$$

We choose t as the minimum of the both t's in (5) and (6).
Next we choose s:
Since d_1 is real analytic in a neighborhood of S_0' and harmonic on the fibers we can, for an appropriate chosen Y, apply (1) with $C = S_0'$ and $\nu = 0$ and get: There exists an $s > 0$ s.th.

$$\|d_1\|_{S',s} < \infty \tag{7}$$

We have $w = g(\tilde{w},\tilde{z})$ and $\tilde{w} = f(w,z)$ in $U(\partial X)$ with real analytic and on the fibers holomorphic functions f and g. Since the z–free summand of the Taylor series expansion always vanishes we can, with the same argument as for the proof of (7), choose s so small that

$$\|g(\tilde{w},\tilde{z}) - g(\tilde{w},0)\|_{\tilde{S''},s} < t \;,\; \|f(w,z) - f(w,0)\|_{S',s} < t \tag{8}$$

Choose s as the minimum of those s in (7) and (8).
So we now have fixed numbers s, t, r, δ, ρ with the properties above which we leave fixed for the rest of this proof.
4) We want to prove:

$$\|d_n\|_{S',s} \le \rho^{n-1}\|d_1\|_{S',s} < \infty \; for\; n \in I\!N \tag{9}$$

The second inequality is (7). Assume that (9) is true for n. Then we are ready if we have proved:

$$\|e_n\|_{\tilde{S''},s} \le (1+r)\|d_n\|_{S',s} \;;\; \|d_{n+1}\|_{S',s} \le \delta(1+r)\|e_n\|_{\tilde{S''},s} \tag{10}$$

We only prove the second assertion of (10), the first can be proved in the same way. If $d_{n+1} = \sum_{\mu\in\mathbb{Z}}\sum_{\nu\in I\!N^2} \tilde{a}_{\nu\mu}(\tilde{w}_0,0)(\tilde{w}-\tilde{w}_0)^\mu \tilde{z}^{\nu_1}\overline{\tilde{z}}^{\nu_2}$ around $(\tilde{w}_0,0) \in S_0'$ we have:

$$\|d_{n+1}\|_{S',s} = \|\sum_{\mu\in\mathbb{Z}} (f(w,z) - f(w_0,0))^\mu (\sum_{\nu\in I\!N^2} \tilde{a}_{\nu\mu}(\tilde{w}_0,0) z^{\nu_1}\overline{z}^{\nu_2})\|_{S',s} =$$

$$= \|\sum_{\mu\in\mathbb{Z}} (f(w_0,z) - f(w_0,0))^\mu (\sum_{\nu\in I\!N^2} \tilde{a}_{\nu\mu}(\tilde{w}_0,0) z^{\nu_1}\overline{z}^{\nu_2})\|_{S',s} \le (\|\cdot\|_{S',s}\; is\; submultipli-$$

$$cative) \le \sum_{\mu\in\mathbb{Z}} ((\|f(w_0,z) - f(w_0,0)\|_{S',s})^{|\mu|} \|\sum_{\nu\in I\!N^2} \tilde{a}_{\nu\mu}(\tilde{w}_0,0) z^{\nu_1}\overline{z}^{\nu_2})\|_{S',s}) \le$$

$$(apply\; (8)) \le \sum_{\mu\in\mathbb{Z}} t^{|\mu|} (\sum_{\nu\in I\!N^2} s^{\nu_1+\nu_2} \sup_{(\tilde{w}_0,0)\in S_0'} |\tilde{a}_{\nu\mu}(\tilde{w}_0,0)|) =$$

$$= \sum_{\nu\in I\!N^2} s^{\nu_1+\nu_2} (\sum_{\mu\in\mathbb{Z}} t^{|\mu|} \sup_{(\tilde{w}_0,0)\in S_0'} |\tilde{a}_{\nu\mu}(\tilde{w}_0,0)|) < (apply(5))$$

$$< \sum_{\nu\in I\!N^2} s^{\nu_1+\nu_2}\delta(1+r) \sup_{(\tilde{w}_0,0)\in S_0''} |\frac{\partial^\nu}{\partial\tilde{z}^{\nu_1}\overline{\tilde{z}}^{\nu_2}} \frac{1}{\nu_1!\nu_2!} d_{n+1}(\tilde{w}_0,0)| =$$

$$= \sum_{\nu\in I\!N^2} s^{\nu_1+\nu_2}\delta(1+r) \sup_{(\tilde{w}_0,0)\in S_0''} |\frac{\partial^\nu}{\partial\tilde{z}^{\nu_1}\overline{\tilde{z}}^{\nu_2}} \frac{1}{\nu_1!\nu_2!} e_n(g(\tilde{w}_0,0),0)| =$$

$$= \delta(1+r)\|e_n\|_{S'',s}^{\sim}$$

5) We have $\sum_{n\in I\!N} d_n = q + \log|w| - q_1''$ on $\overline{X''}$. With the same argument as in [GR,p.16] and by using that the space of the continuous complex functions on S_0' with the maximum norm is complete we can prove that 0–locally we have $d(w,z) = \sum_{\nu\in I\!N^2} a_\nu(w) z^{\nu_1}\overline{z}^{\nu_2}$ for $(w,z)\in S'$ and $|z|<s$ with on S_0' continuous functions $a_\nu(w)$ and with

$$\|d\|_{S',s} = \lim_{m\to\infty} \|\sum_{n\le m} d_n\|_{S',s} \le \frac{1}{1-\rho}\|d_1\|_{S',s}$$

Further we have for $u,r\in I\!R^+$ with $r+u<s$:

$$\sup_{(w,z)\in S',\,|z|<r} |\frac{\partial^{\mu_1+\mu_2} d}{\partial z^{\mu_1}\overline{z}^{\mu_2}}| = \sup_{(w,z)\in S',\,|z|<r} |\sum_{\nu_i\ge\mu_i} \mu_1!\begin{pmatrix}\nu_1\\ \mu_1\end{pmatrix}\mu_2!\begin{pmatrix}\nu_2\\ \mu_2\end{pmatrix} a_\nu(w) z^{\nu_1-\mu_1}\overline{z}^{\nu_2-\mu_2}| \le$$

$$\le \mu_1!\mu_2!(\frac{1}{u})^{\mu_1+\mu_2}\sum_{\nu_i\ge\mu_i}\sup_{(w,0)\in S_0'}|a_\nu(w)|s^{\nu_1+\nu_2}[\begin{pmatrix}\nu_1\\ \mu_1\end{pmatrix}(\frac{r}{s})^{\nu_1-\mu_1}(\frac{u}{s})^{\mu_1}\begin{pmatrix}\nu_2\\ \mu_2\end{pmatrix}(\frac{r}{s})^{\nu_2-\mu_2}(\frac{u}{s})^{\mu_2}] \le$$

$$\le \mu_1!\mu_2!(\frac{1}{u})^{\mu_1+\mu_2}\frac{1}{1-\rho}\|d_1\|_{S',s}$$

since

$$\sum_{\mu_i=0}^{\nu_i}\begin{pmatrix}\nu_i\\ \mu_i\end{pmatrix}(\frac{r}{s})^{\nu_i-\mu_i}(\frac{u}{s})^{\mu_i} = (\frac{r}{s}+\frac{u}{s})^{\nu_i} < 1$$

So the proof of real analyticity of q is complete.
6) The function q is real analytic and harmonic in w, hence q_w is holomorphic in w and it follows from the Cauchy integral formula that for every cycle Γ lying in a fiber we have $\int_\Gamma h_w dw = 0$ and $\int_\Gamma h_{\overline{w}} d\overline{w} = 0$. We may assume that 0–locally there exists a w_0 s.th. all points (w_0,z) lie in X. Hence if $\Gamma(w_0,w,z)$ denotes a cycle lying in the z–fiber connecting (w_0,z) and (w,z) and if we define $t(w,z) = i\int_{\Gamma(w_0,w,z)}(h_{\overline{w}}d\overline{w} - h_w dw)$ the function $q+it$ is holomorphic in w and real analytic, the latter can be seen easily by choosing the cycle $\Gamma(w_0,w,z)$ in an appropriate way. We put $\dot{F}(w,z) = e^{q+it}w$ and have $\dot{F}_w(0,z)\neq 0$, $|\dot{F}|\equiv 1$ on ∂X and $\dot{F}$ has only $w=0$ for zero. Hence on every fiber $\dot{F}$ yields a one sheeted analytic covering of Δ and hence is biholomorphic there. So we have got a fiber preserving, bi real analytic and on the fibers holomorphic map from X to $\Delta\times D$ which maps s onto the zero section. Since with Lemma 2.2 F can be extended topologically into the borders the proof is complete. □

As a consequence we have:

Lemma 2.5 *Let* $ds^2 = g(w,z)dwd\overline{w}$ *be the hyperbolic metric (c.f. subsect. 1.3) on the fiber* X_z. *Then* $g(w,z)$ *is a real analytic function.*

Proof: By the last Lemma the fibration X is z–locally real analytically equivalent with $\Delta\times D$ through F. If we bring back the metric on X through F to $\Delta\times D$, it is constant in z and real analytic in w. So $g(w,z)$ is real analytic. □

4. In all what follows in this paper we only consider two cases for the fibration $X\subset \mathbb{C}\times D$ with fixed holomorphic cross section $s: D\to X$:
case 1: X has the properties 1) to 3) of subsect. 1.1 and the function $\tilde{w} = f(w,z)$ of 2) is holomorphic.
case 2: X has the properties 1) to 3) of subsect. 1.2.
We put $\check{D}(z_0) = \{z = u+iv\in D : u = \Re(z_0)\}$ and prove:

Lemma 2.6 *Assume that* $X \subset \mathbb{C} \times D$ *has the properties of case 1 or of case 2. Then z–locally the fibration X is biholomorphically (and fiber preserving) equivalent with a new fibration X having the properties of case 2 and the properties* $X|_{\check{D}(z)} = \Delta \times \check{D}(z)$, $s = 0 \times D$. *The equivalence map can be extended topologically into the borders.*

Proof: We again assume $z = 0$ and denote $\check{D} := \check{D}(0)$. We only need to prove the first assertion, since then the second one follows from Lemma 2.2 and subsect. 1.2. By restriction of the map F of Lemma 2.4 to $\check{D}$ we obtain 0–locally a topological fiber preserving map $\check{F} : \overline{X}|_{\check{D}} \to \overline{\Delta} \times \check{D}$ with the following properties:
1) $s|_{\check{D}}$ is mapped onto $0 \times \check{D}$
2) Each fiber X_z, $z \in \check{D}$ is mapped biholomorphically onto $\Delta \times \{z\}$
3) $\check{F}|_{(X|_{\check{D}})}$ is bi real analytic.
The second case is easy. If we apply Lemma 2.1 with $B = \check{D}$ we get that 0–locally $\check{F}$ can be extended into a neighborhood V of $\overline{X}|_{\check{D}}$ in $\mathbb{C} \times \check{D}$ s.th. it is real analytic and biholomorphic in each fiber. This extension can canonically be extended to a holomorphic map $\tilde{F} : W \to \mathbb{C} \times \tilde{U}$ where $\tilde{U}(0)$ denotes a small neighborhood of 0 in D and W is a neighborhood of $\overline{X} \cap \pi^{-1}(\tilde{U})$. Now since $\tilde{F}$ was biholomorphic in $V \cap \pi^{-1}(0)$, we can show exactly as in the proof of Lemma 2.1. that $\tilde{U}$ and V can be chosen so small that $\tilde{F}$ is bijective. At last we can achieve $s = 0 \times D$ by the transformation $w' = w - s(z)$.
The first case is not so easy because we can't extend $\check{F}$ holomorphically over the edges of $\overline{X}|_{\check{D}}$. So we first have to look for a way to get ∂X smooth. We proceed like in the proof of Lemma 2.2 and get a manifold $\check{Y}$ consisting of X with coordinates (w, z) and of $(\mathbb{C} \times D) \setminus (X \setminus U)$ with coordinates (w^*, z) in which $\partial X = \partial\Delta \times D$. Now we can use Lemma 2.1 to $F|_{X \cap U}$ in the coordinate system $(\mathbb{C} \times D) \setminus (X \setminus U)$. So we get the desired extension of X, but in $\check{Y}$ and not in $\mathbb{C} \times D$. We now can restrict this extension to $\check{D}$ and then extend canonically to a holomorphic map $\tilde{F} : W \to \mathbb{C} \times \tilde{U}$ with W and $\tilde{U}$ like in the first case, and even the bijectivity argument of the proof of Lemma 2.1 works, so W can be chosen so small that $\tilde{F}$ becomes biholomorphic. If we now restrict $\tilde{F}$ to X and bring s to $0 \times D$ X has the desired properties. □

3 The Infinitesimal Deformation with respect to a Holomorphic Cross Section

In the subsects. 3.1 and 3.2 we introduce the notions of infinitesimal deformation and of a square integral of infinitesimal deformation, both in resp. of the given holomorphic cross section s, for smooth families X of bounded domains with the properties of case 2 of subsect. 2.4. In subsect. 3.3 we do this for X with the properties of case 1 of subsect. 2.4. In subsect. 3.4 we describe how the square integral can be computed.

1. In every point of ∂X there exists a unique complex tangent (also c.f. subsect. 3.4). We will prove:
On every fiber $\overline{X_z}$ we can construct a vector field $\xi_z(w)$ in $\mathbb{C} \times D$ with the following properties:
1) $\xi_z(w)$ is mapped on $\partial/\partial z$ by the canonical projection π (c.f. subsect. 1.1).
2) $\xi_z(w)|_{\partial X_z}$ lies in the complex tangent at ∂X.
3) $\xi_z(s(z))$ lies in direction of $s(D)$ in X.
4) $\xi_z(w)$ is differentiable in w and real analytic in a neighborhood of ∂X_z in $\overline{X_z}$.
We even can choose $\xi_z(w)$ for every $z \in D$ in such a way that $\xi(w, z) := \xi_z(w)$ is

differentiable in $\overline{X}$ and real analytic in a neighborhood of ∂X in $\overline{X}$. If Ω_z denotes the set of all differentiable vector fields $\omega_z(w)$ in $\mathbb{C} \times D$ defined on $\overline{X_z}$ in direction of $\overline{X_z}$ with $\omega_z|_{\partial X_z} = 0$ and $\omega_z(s(z)) = 0$ which are real analytic in a neighborhood of ∂X_z in $\overline{X_z}$ a field $\xi_z(w)$ with the properties 1) to 4) is uniquely determined up to an $\omega_z(w) \in \Omega_z$.
It is a simple computation to prove that in every point of ∂X and in $s(D)$ $\xi(w,z)$ exists, is uniquely determined there and real analytic on ∂X. We can extend $\xi(w,z)$ real analytically into a neighborhood $V'(\partial X) \cup V''(s(D)) \subset \overline{X}$ by standart methods. Then we can extend $\xi(w,z)$ differentiably to $\overline{X}$ by a partition of the unity in such a way that there exists a relative compact subneighborhood $V(\partial X) \subset V'$ s.th. $\xi(w,z)$ isn't changed in $V(\partial X) \cup s(D)$ and hence has all desired properties. The uniqueness of $\xi_z(w)$ up to elements of Ω_z is evident.
Let $Z(X_z)$ be the set of all differentiable (0,1)–forms on $\overline{X_z}$ with coefficients in the tangent vectors of $\overline{X_z}$ and $B(X_z) = \{\overline{\partial}\omega_z : \omega_z \in \Omega_z\}$, where we identify canonically tangent vectors at $\overline{X_z}$ and vectors in $\mathbb{C} \times D$ in direction of $\overline{X_z}$ and where we identify X_z and X'_z, where X and X' are two different fibrations, only then if z–locally there exists a biholomorphic fiber preserving map from a neighborhood of $\overline{X}$ to a neighborhood of $\overline{X'}$. We have $B(X_z) \subset Z(X_z)$ and denote by $H(X_z)$ the cohomology group $Z(X_z)/B(X_z)$.
If (written holomorphically) $\xi_z(w) = \xi^o_z(w)\frac{\partial}{\partial w} + \frac{\partial}{\partial z}$ we set $\overline{\partial}\xi_z(w) = ((\xi^o_z)_{\overline{w}}(w))\frac{\partial}{\partial w} d\overline{w}$, this is invariant under z–locally fiber preserving holomorphic coordinate transformations defined in a neighborhood of $\overline{X}$. So $\overline{\partial}\xi_z(w)$ is an element of $Z(X_z)$. Since $\xi_z(w)$ is uniquely determined up to an element of Ω_z we have that $\rho(\overline{X}, s, z) := \overline{\partial}\xi_z(w)/B(X_z) \in H(X_z)$ is well defined.

Definition 3.1 $\rho(\overline{X}, s, z)$ *is called the infinitesimal deformation with resp. to s.*

We already have proved the following

Proposition 3.2 $\rho(\overline{X}, s, z)$ *is independant of z–local fiber preserving holomorphic coordinate transformations which are defined in a neighborhood of* $\overline{X}$.

Definition 3.3 $\overline{X}$ *is called stable in z with resp. to s if there exists a coordinate transformation like in Prop. 3.2 s.th. in the new coordinates we have* $X|_{DP_z} = X_z \times DP_z$ *and* $s|_{DP_z} = s(z) \times DP_z$, *where* DP_{z_0} *denotes the double point in* z_0 *(that means that we have* X_{z_0} *with the structure sheaf* $\mathcal{O}_{\mathbb{C}\times D}/(z-z_0)^2\mathcal{O}_{\mathbb{C}\times D}$*).*

Remark: $X|_{DP_z} = X_z \times DP_z$ and $s|_{DP_z} = s(z) \times DP_z$ is equivalent with $\xi_z = \frac{\partial}{\partial z}$ on $\partial X_z \cup \{s(z)\}$.

Proposition 3.4 $(\overline{X}, s)$ *is stable in z iff* $\rho(\overline{X}, s, z) = 0$.

Proof: Assume $\rho(\overline{X}, s, z_0) = 0$. Then we can find a $\xi_{z_0} = \xi^o_{z_0}\frac{\partial}{\partial w} + \frac{\partial}{\partial z}$ with $\overline{\partial}\xi_{z_0} = 0$, hence $\xi^o_{z_0}(w)$ is holomorphic. Since ∂X_{z_0} is real analytically smooth we can extend $\xi^o_{z_0}$ real analytically into a neighborhood of ∂X_{z_0} in $\mathbb{C} \times \{z_0\}$, and since $\overline{\partial}\xi^o_{z_0}$ is identically zero in $\overline{X_z}$ it is zero in the neighborhood by the identity lemma for power series, where we denote the extension again by $\xi^o_{z_0}$. Hence $\xi^o_{z_0}(w)$ is in a neighborhood of $\overline{X_{z_0}}$ holomorphic and

$$w^* = w - (z - z_0)\xi^o_{z_0}(w)\,,\ z^* = z$$

z–locally gives a fiber preserving holomorphic mapping defined in a neighborhood of $\overline{X}$. This map is locally biholomorphic in every point of $\overline{X_{z_0}}$ in resp. of $\mathbb{C} \times D$ and hence biholomorphic if the neighborhoods are chosen small enough. It is easily computed in this coordinates (w^*, z^*) that $\overline{X}$ is stable in z_0 in resp. of s.
The opposite direction is evident. □

2. First we introduce a standart vector field γ_s in $\overline{X}$:
We proved in Lemma 2.4 that there z–locally is a fiber preserving topological map $F:\overline{X}\to\overline{\Delta}\times D$ which is bi real analytic in the interior and holomorphic on the fibers, and maps s onto the zero section. With Lemma 2.1 we have that F is still real analytic on $\overline{X}$ since ∂X is real analytic. We denote the coordinates in $\overline{\Delta}\times D$ by (w',z'). The map F is uniquely determined up to a rotation of the fibers which depends real analytically on z'. The inverse image of the holomorphic vector field $w'\frac{\partial}{\partial w'}$ on $\overline{\Delta}\times D$ in $\overline{X}$ is real analytic and holomorphic on the fibers there. It has a zero of the first order in $s(D)$, is different from zero elsewhere and is vertical to ∂X on ∂X. We call this field the *standart vector field* γ_s *with resp. to* s. It is defined independantly of F.

Definition 3.5 *A (global resp. z–local) vector field η on $\overline{X}$ in direction of the fibers is called s–harmonic if there (globally resp. z–locally) exists a continuous function $f:\overline{X}\to\mathbb{C}$ which is harmonic in every fiber X_z s.th. $\eta=f\cdot\gamma_s$. If $z\in D$ then a vector field η on $\overline{X_z}$ in direction of X_z is called s–harmonic if there exists a continuous function $f:\overline{X_z}\to\mathbb{C}$ which is harmonic in X_z and $\eta=f\cdot\gamma_s$ holds.*

Lemma 3.6 *Let $\underline{\eta}$ be a real analytic vector field defined on ∂X. Then there exists a uniquely determined s–harmonic vector field η on $\overline{X}$ which is real analytic in X, real analytic in every fiber $\overline{X_z}$ and which extends $\underline{\eta}$ from ∂X to $\overline{X}$. The same is true z–locally in $\overline{X}$ or for a single fiber $\overline{X_z}$.*

Proof: We only have to prove that z–locally. With Lemma 2.4 and Lemma 2.1 there z–locally exists a fiber preserving, bi real analytic and on the fibers holomorphic mapping $F:X\to\Delta\times D$ which still is defined and bi real analytic in a neighborhood of $\overline{X}$. Now we can lift back the problem to $\overline{\Delta}\times D$ where we can solve it with Lemma 2.3. This proves our assertion if we additionally show:
If $b:\partial\Delta\to\mathbb{C}$ is real analytic, then the harmonic continuation $f:\overline{\Delta}\to\mathbb{C}$ can be extended real analytically into a neighborhood of $\overline{\Delta}$ in $\mathbb{C}$.
If we devellop b into a Fourier series we get on $\partial\Delta$:

$$b(w)=a_0+\sum_{n>0}a_nw^n+\sum_{n>0}b_n\overline{w}^n \tag{11}$$

This series converges absolutely uniformly on $\partial\Delta$ and is uniquely determined.
Since $b(w)$ is real analytic on $\partial\Delta$, we can extend it by standart technics (c.f. [BS,p.373]) to a holomorphic function $h:R\to\mathbb{C}$, where R is a ring of circles with $\partial\Delta\subset\subset R$. Then we can devellop h into a Laurent series:

$$h(w)=\sum_{n\in\mathbb{Z}}e_nw^n \; in \; R$$

On $\partial\Delta$ we have

$$b(w)=\sum_{n\in\mathbb{Z}}e_nw^n=e_0+\sum_{n>0}e_nw^n+\sum_{n>0}e_{-n}\overline{w}^n$$

hence $e_n=a_n$, $n\geq 0$. So $\sum_{n>0}a_n$ has a convergence radius greater than 1.
If we apply the same argument on $\overline{b}$ instead of on b, we get that $\sum_{n>0}\overline{b_n}w^n$ and hence $\sum_{n>0}b_n\overline{w}^n$ has a convergence radius greater than 1. So this also is true for the harmonic continuation of b (c.f. (11)). □

To define the concept of a square integral of infinitesimal deformation in resp. of s we first make a holomorphic coordinate transformation defined in a neighborhood of $\overline{X}$ in $\mathbb{C}\times D$ which brings s to the zero section $0\times D$ (e.g. $w'=w-s(z)$, $z'=z$). The new coordinates

are again denoted with (w,z).
Then we apply Lemma 3.6 on $\underline{\eta} = (\xi(w,z) - \partial/\partial z)|_{\partial X}$, where we let $\xi(w,z)$ be like in subsect. 3.1. With this Lemma and Definition 3.5 we get a uniquely determined function $f(w,z) : \overline{X} \to \mathbb{C}$ with $(f \cdot \gamma_s)|_{\partial X} = \underline{\eta}$. With this function we define:

Definition 3.7 $r_s(z) = -i \int_{X_z} \overline{\partial} f \wedge \partial \overline{f}$ *is called the square integral of infinitesimal deformation in resp. to s.*

We have to show that $r_s(z)$ is coordinate invariant, especially that it doesn't depend on the holomorphic coordinate transformation which has brought s to the zero section. So we prove:

Proposition 3.8 $r_s(z)$ *is invariant under z–locally fiber preserving holomorphic coordinate transformations* $T : w' = t(w,z)$, $z' = z$ *of X which map the zero section onto itself and for which in both coordinates* ∂X *is real analytically smooth.*

Proof: First we can with Lemma 2.1 z–locally extend T with all its properties into a neighborhood of $\overline{X}$. If we make such a coordinate transformation then we have $T_*(\xi_z) = (t_w \xi_z^o + t_z)\frac{\partial}{\partial w'} + \frac{\partial}{\partial z'}$ and $t(0,z) \equiv 0$ hence $t_z(0,z) \equiv 0$. So there exists a holomorphic function f_1 with $t_z \frac{\partial}{\partial w'} = f_1 \cdot \gamma_s$ and $\eta' = (f + f_1) \cdot \gamma_s$ is the new continuation from the boundary if $\eta = f \cdot \gamma_s$ was the old one. Since we have $\overline{\partial}(f + f_1) = \overline{\partial} f$ our assertion is proved. □

Remark: 1) The connection between $\rho(\overline{X}, s, z)$ and $r_s(z)$ is the following: If we have brought s to $0 \times D$, then $f \cdot \gamma_s + \frac{\partial}{\partial z}$ is one possible field $\xi_z(w)$ (like defined in subsect. 3.1) which by its special properties is uniquely determined. We need such a uniquely determined field because otherwise the square integral over the $\overline{\partial}$–derivative would depend on the special choice of the field. The standart vector field γ_s is needed to define the concept of a "harmonic vector field" coordinate invariant.
2) With the following Theorem 3.9 and Prop. 3.4 we have:

$$\rho(\overline{X}, s, z) = 0 \quad iff \quad r_s(z) = 0$$

Of course, this can also be proved directly.
3) We have $\rho(\overline{X}, s, z) = 0$ (and hence $r_s(z) = 0$) iff the field $\xi_z(w) - \frac{\partial}{\partial z}$ on ∂X_z, i.e. the component which is lying in the fiber of that vector of the complex tangent which is lying over $\frac{\partial}{\partial z}$, can holomorphically be extended to a tangent vector field $\omega(w)$ in $\overline{X_z}$ with $\omega(s(z)) = \frac{\partial s(z)}{\partial z} \frac{\partial}{\partial w}$.

The function $r_s(z)$ has important properties:

Theorem 3.9 $\overline{X}$ *is stable with resp. to s in* $z \in D$ *iff* $r_s(z) = 0$.

Proof: We assume that $r_s(z_0) = 0$. Then we may assume that s is the zero section and we have an in $\overline{X_{z_0}}$ continuous and in X_{z_0} holomorphic function f with $f \cdot \gamma_s = \xi_{z_0} - \frac{\partial}{\partial z}$ on ∂X_{z_0}, which with Lemma 3.6 can be extended real analytically and hence holomorphically into a neighborhood of $\overline{X_{z_0}}$. So

$$w^* = w - (z - z_0)\eta^o(w)\,,\ z^* = z$$

where $\eta^o \frac{\partial}{\partial w} = \eta$, gives z_0–locally a fiber preserving biholomorphic map defined in a neighborhood of $\overline{X}$. In these new coordinates we have $\xi_{z_0} = \frac{\partial}{\partial z}$ on ∂X_{z_0} and $\xi_{z_0}(0) = \frac{\partial}{\partial z}$ since $\eta^o(0) = 0$. So $(\overline{X}, s)$ is stable in z_0. Since the opposite direction again is evident the proof is comlete. □

Definition 3.10 $\overline{X}$ *is said to be z–locally a cartesian product with resp. to s if there z–locally exist holomorphic coordinates defined in a neighborhood of $\overline{X}$ s.th. in those coordinates ∂X and s are cartesian products.*

Theorem 3.11
a) Assume that X has the properties of case 2 <u>and</u> of case 1 of subsect. 2.4. Then we z_0–locally have $r_s(z) \equiv 0$ iff $\overline{X}$ is z_0–locally a cartesian product with resp. to s.
b) If X only has the properties of case 2 of subsect. 2.4 the assertion a) is wrong.

Proof:
1) We only have to prove one direction of a), the other is evident. We may assume $s = 0 \times D$. To simplify writing we assume $z_0 = 0$. With Definition 3.5 and Lemma 3.6 and since 0–locally we have $r_s(z) \equiv 0$ there 0–locally exists a continuous function $f : \overline{X} \to \mathbb{C}$ which is real analytic in X and holomorphic on the fibers s.th. for $\eta = f \cdot \gamma_s$ we have $\eta = \xi_z - \frac{\partial}{\partial z}$ on ∂X. Since $\gamma_s(0, z) \equiv 0$ the same holds for η. From Lemma 3.6 we know that in every fiber X_z $f|_{X_z}$ can holomorphically be extended into a neighborhood of $\overline{X_z}$ in $\mathbb{C} \times \{z\}$. By assumption there 0–locally exists a neighborhood $U(\partial X)$ with holomorphic coordinates $(\tilde{w}, \tilde{z})$ s.th. in those coordinates ∂X is a cartesian product, so $\frac{\partial}{\partial \tilde{z}}$ is the complex tangent vector over $\frac{\partial}{\partial z}$ (in D). Since $\eta + \frac{\partial}{\partial z}|_{\partial X}$ also was lying in the complex tangent over $\frac{\partial}{\partial z}$ (in D) this field gives (in $(\tilde{w}, \tilde{z})$–coordinates) a vector field which in every fiber is holomorphic, can fiberwise be extended into a neighborhood of the closure of the fiber, and coincides with the field $\frac{\partial}{\partial \tilde{z}}$ on ∂X. By the identity lemma for holomorphic functions it follows that on every fiber the both fields coincide. Now the field $\frac{\partial}{\partial \tilde{z}}$ is holomorphic in $U(\partial X)$ and extends $\eta + \frac{\partial}{\partial z}$. So the function f can be holomorphically extended s.th. the field $\eta + \frac{\partial}{\partial z}$ is defined in $X \cup U(\partial X)$ and holomorphic in $U(\partial X)$. Since f is real analytic in X it is a consequence of the identity lemma for power series that $\eta + \frac{\partial}{\partial z}$ and hence η also is holomorphic in X. So we now 0–locally have constructed a holomorphic vector field η on $U(\partial X) \cup X$ in direction of the fibers with $\eta(0, z) \equiv 0$ for which $\eta + \frac{\partial}{\partial z}$ is in $(\tilde{w}, \tilde{z})$–coordinates the field $\frac{\partial}{\partial \tilde{z}}$.
2) We may assume that $\tilde{w}(w, 0) \equiv w$ on $U(\partial X)_0$ since we can extend the biholomorphical map $w(\tilde{w}, 0)$ independantly of z to the other fibers of $U(\partial X)$ in $(\tilde{w}, \tilde{z})$–coordinates. Under this transformation ∂X remains a cartesian product and the field $\frac{\partial}{\partial \tilde{z}}$ isn't changed. We further 0–locally may assume that $X_0 = \Delta$ (c.f. subsect. 1.2). Now 0–locally there exists a ring of circles R in $U(\partial X)_0$ with $\partial X_0 \subset\subset R$ and an $\epsilon > 0$ s.th $P := R \times \{|z| < \epsilon\} \subset\subset U(\partial X)$,

$$\tilde{w}(w, z) = \sum_{n \geq 0} f_n(w)\, z^n \text{ in } P \tag{12}$$

with in R holomorphic functions $f_n(w)$ and s.th.

$$\sum_{n \geq 0} \sup_{w \in R} |f_n(w)| \cdot \epsilon^n < \infty \tag{13}$$

the last statement is an application of Lemma 2.3 for an appropriate chosen Y. If we now can show that every $f_n(w)$ can holomorphically be extended into $R \cup X_0$, the series $\sum_{n \geq 0} f_n(w) \cdot z^n$ converges uniformly by the maximum principle and (13) and hence converges against a holomorphic function. So we have extended $\tilde{w}(w, z)$ into the (0–local) neighborhood $(R \cup X_0) \times \{|z| < \epsilon\}$ of X. Since $\tilde{w}(w, 0) \equiv w$ in $U(\partial X)_0$ and hence in $R \cup X_0$ the functional determinant of the mapping $(\tilde{w}(w, z), z)$ is identical 1, so $(\tilde{w}(w, z), z)$ 0–locally is biholomorphic (c.f. the proof of Lemma 2.1) and hence gives holomorphic coordinates in which ∂X is a cartesian product. If we further show that $f_n(0) = 0$ for every $n \geq 0$ s again is the zero section in the new coordinates, hence also a cartesian product.
3) We still have to show that the holomorphic functions f_n can holomorphically be extended

from R to $R \cup X_0$ with $f_n(0) = 0$. Since we know from 1) that the field $\eta + \frac{\partial}{\partial z}$ (in (w, z)–coordinates) becomes $\frac{\partial}{\partial \tilde{z}}$ (in $(\tilde{w}, \tilde{z})$–coordinates) we have

$$(\frac{\partial}{\partial w}\tilde{w}(w, z)) \cdot \eta(w, z) + \frac{\partial}{\partial z}\tilde{w}(w, z) \equiv 0 \; in \; U(\partial X) \tag{14}$$

$f_0(w)$ can be extended from R to $R \cup X_0$ with $f_0(0) = 0$ since by 2) we have $f_0(w) = \tilde{w}(w, 0) \equiv w$ on R. Assume now that $f_0, ..., f_n$ can be holomorphically extended from R to $R \cup X_0$. Then we have in R:

$$f_{n+1}(w) = \frac{1}{(n+1)!}\frac{\partial^{n+1}}{\partial z^{n+1}}\tilde{w}(w, z)|_{z=0} = \frac{1}{(n+1)!}\frac{\partial^n}{\partial z^n}((-\frac{\partial}{\partial w}\tilde{w}(w, z)) \cdot \eta) =$$

$$= \frac{1}{(n+1)!} - \sum_{\nu=0}^{n} \binom{n}{\nu} (\frac{\partial^{n-\nu}}{\partial z^{n-\nu}}(\frac{\partial}{\partial w}\tilde{w}(w, z)))(\frac{\partial^\nu}{\partial z^\nu}\eta(w, z))|_{z=0} =$$

$$= -\frac{1}{n+1}\sum_{\nu=0}^{n}\frac{1}{\nu!}(\frac{\partial}{\partial w}f_{n-\nu}(w))(\frac{\partial^\nu}{\partial z^\nu}\eta(w, z))|_{z=0}$$

By our assumptions the last term can holomorphically be extended to $R \cup X_0$ and since $\eta(0, z) \equiv 0$ we have $f_{n+1}(0) = 0$. This completes the proof of a).
4) We give a counterexample:

$$D = \Delta\,, \; X = \{(w, z) \in \mathbb{C} \times \Delta : \; w\overline{w} + z\overline{z} < 1\}\,, \; s = 0 \times \Delta$$

It is an easy computation that X is of case 2 of subsect. 2.4. With a formula for the vector of the complex tangent lying over $\frac{\partial}{\partial z}$ (in $z \in D$) which we will devellop in subsect. 3.4 we will get for $(w, z) \in \partial X$:

$$\xi_z(w) = -\frac{\overline{z}}{\overline{w}}\frac{\partial}{\partial w} + \frac{\partial}{\partial z} \tag{15}$$

Further we have: $X_z = \sqrt{1 - |z|^2} \cdot \Delta$, $\gamma_s|_{X_z} = \sqrt{1 - |z|^2} \cdot w \cdot \frac{\partial}{\partial w}$, hence $f(w; z) = -\frac{\overline{z}}{1-|z|^2}$ is the s–harmonic continuation. So we have $r_s(z) \equiv 0$ on Δ.
The Levi form of ∂X is $L(w, z) = dwd\overline{w} + dzd\overline{z}$, so we have $L(w, z)(\xi_z(w)) > 0$. From this it follows that there doesn't exist a $z \in \Delta$ s.th. z–locally X is a cartesian product, since then the Levi form would have to vanish on the complex tangent. □

Up to now we only have dealt with *one* holomorphic cross section s in X. Now we want to see how the square integral over the infinitesimal deformation changes if the holomorphic cross section is changed. First we define a local version of such a holomorphic cross section.

Definition 3.12 *Let X be like in subsect. 1.1, $z \in D$, $U(z) \subset D$ a neighborhood and $s : U(z) \to X$ a holomorphic cross section. Then s is called a z–local holomorphic cross section in X.*

For such a z–local holomorphic cross section s in X the square integral $r_s(z)$ of the infinitesimal deformation is well defined.
We assume again that X is of case 2 of subsect. 2.4, and we further assume two 0–local holomorphic cross sections $s(z)$ and $t(z)$ in X. We want to compare $r_s(0)$ and $r_t(0)$. With Prop. 3.8, Lemma 2.6 and Lemma 2.1 we may assume that $X_0 = \Delta$ and that s is the zero section since this situation can 0–locally be achieved by a coordinate transformation which doesn't change $r_s(0)$ and $r_t(0)$. Then we have:

Theorem 3.13
a) If $t(0)=0$ *and* $r_s(0)=-i\int_\Delta |f_{\overline{w}}|^2 d\overline{w}\wedge dw$ *, we have* $r_t(0)=-i\int_\Delta |f_{\overline{w}}-t_z(0)|^2 d\overline{w}\wedge dw$ *. Especially we have* $r_t(0)=0$ *at most for one complex number* $t_z(0)$ *and* $r_s(0)=r_t(0)$ *if* $t_z(0)=0$ *.*
b) $r_s(0)$ *only depends on* $X|_{DP_0}$ *and* $s|_{DP_0}$ *, where* DP_0 *again denotes the double point over* $0\in D$ *.*

Proof: Assume $t(0)=0$. Then $w'=w-t(z)$, $z'=z$ is a holomorphic coordinate transformation. If $\xi_0=\xi_0^o(w)\frac{\partial}{\partial w}$ we have $\xi_0=(\xi_0^o(w)-t_z(0))\frac{\partial}{\partial w'}+\frac{\partial}{\partial z'}$ and hence on ∂X_0:

$$\xi_0-\frac{\partial}{\partial z'}=\xi_0^o(w)\frac{\partial}{\partial w'}-t_z(0)\frac{\partial}{\partial w'}=f\cdot w\cdot\frac{\partial}{\partial w'}-t_z(0)\overline{w'}w'\frac{\partial}{\partial w'}=$$

$$f\cdot w'\cdot\frac{\partial}{\partial w'}-t_z(0)\overline{w'}\gamma_t=(f-t_z(0)\overline{w'})\gamma_t$$

From this assertion a) follows immediately.
Assertion b) is a consequence of assertion a) and the definition of $r_s(z)$. □

If $t(0)\neq s(0)$ $r_s(0)$ and $r_t(0)$ are not related so closely since then $\gamma_s|_{X_0}\neq\gamma_t|_{X_0}$. To give an idea of this we prove a proposition which is interesting for some applications, e.g. for those in the next section.

Proposition 3.14 *There exists a smooth fibration* X *with the properties of case 2 of subsect. 2.4 s.th. for every 0–local holomorphic cross section* s *in* X *we have* $r_s(0)>0$ *, but there exists a sequence* s_n *of 0–local holomorphic cross sections in* X *with* $\lim_{n\to\infty} r_{s_n}(0)=0$ *.*

Proof:
1) It is clear that there exists a X with the properties of case 2 of subsect. 2.4 with $X_0=\Delta$, $0\times D\subset X$ and $\xi_0|_{\partial X_0}=(\overline{w}^2-1)w\frac{\partial}{\partial w}+\frac{\partial}{\partial z}$. Assume now that s is the zero section and let t be an arbitrary 0–local holomorphic cross section in X. We set $c:=t(0)$ and define:

$$T_c:\ w'=\frac{w-c}{1-\overline{c}w}\ ,\ z'=z\ ,$$

since $c\in\Delta$ this 0–locally is a holomorphic coordinate transformation. If $f(w'):\overline{\Delta}\to\mathbb{C}$ is the continuous and in Δ harmonic function with $\xi_0-\frac{\partial}{\partial z'}=f\cdot\gamma_t$ on $\partial\Delta$ we have with Theorem 3.13:

$$r_t(0)=-i\int_\Delta |f_{\overline{w'}}-t_{z'}(0)|^2 d\overline{w'}\wedge dw'$$

Hence our first assertion is proved if we have shown that $f_{\overline{w'}}$ cannot be constant.
2) Some elementary computations yield: On $\partial\Delta$ we have (with $w'\overline{w'}=1$):

$$\xi_0-\frac{\partial}{\partial z'}=\frac{(1+\overline{c}w')(1+c\overline{w'})}{(1-c\overline{c})}\Big(\Big(\frac{\overline{w'}+\overline{c}}{1+c\overline{w'}}\Big)^2-1\Big)\gamma_t=$$

$$=\frac{1}{1+c\overline{w'}}(k_1\overline{w'}^2+k_2\overline{w'}+k_3+k_4w')\gamma_t=\sum_{n\geq 0}(-c\overline{w'})^n(k_1\overline{w'}^2+k_2\overline{w'}+k_3+k_4w')\gamma_t=$$

$$=\Big(\frac{k_5}{1+c\overline{w'}}+\frac{k_1}{c}\overline{w'}+\Big(\frac{k_2}{c}-\frac{k_1}{c^2}\Big)+k_4w'\Big)\gamma_t \tag{16}$$

with complex numbers k_i independant of w', especially:

$$k_1=\frac{1-c^2}{1-c\overline{c}}\ ,\ k_5=\frac{(1-c\overline{c})^2}{c^2}$$

Since the last coefficient of the equations (16) is harmonic in Δ it is our f, and we have:

$$f_{\overline{w'}} = \frac{-ck_5}{(1+c\overline{w'})^2} + \frac{k_1}{c} \tag{17}$$

which isn't constant since $k_5 \neq 0$.
3) There exist 0–local holomorphic cross sections s_n in X with $s_n(0) = 1 - \frac{1}{n}$ (in (w,z)–coordinates), s.th. after the coordinate transformation T_{c_n} (with $c_n = s_n(0)$) we have $(s_n)_{z'}(0) = \frac{-k_1(c_n)}{c_n}$. Then with Theorem 3.13 and (17) we have

$$r_{s_n}(0) = -i\int_\Delta |\frac{c_nk_5(c_n)}{(1+c_n\overline{w'})^2}|^2 d\overline{w'} \wedge dw'$$

If $w' = u + iv$ we have $(1 + c_n\overline{w'})(1 + c_nw') \geq (1 + c_nu)^2$, hence

$$r_{s_n}(0) \leq |c_nk_5(c_n)|^2 2\int_\Delta \frac{1}{(1+c_nu)^4} du \wedge dv \leq$$

$$\leq \frac{2}{c_n^2}(1-c_n^2)^4 \int_{-1}^{1} dv \int_{-1}^{1} \frac{1}{(1+c_nu)^4} du = \frac{2}{c_n^2}(1+c_n)^4(1-c_n)^4 \cdot 2 \cdot \frac{1}{3}[\frac{1}{(1-c_n)^3} - \frac{1}{(1+c_n)^3}] =$$

$$= \frac{4}{3}\frac{(1+c_n)^4}{c_n^2}[(1-c_n) - \frac{(1-c_n)^4}{(1+c_n)^3}] \to 0 \ (n \to \infty)$$

This proves the second assertion. □

3. Assume now that X has the properties of case 1 of subsect. 2.4. First we want to define the infinitesimal deformation in a point $z \in D$ with resp. to s and the square integral over it. Let $z \in D$ be arbitrary. Then we know from Lemma 2.6 that z–locally X is biholomorphically equivalent by a map $F : X \to X'$ with a new fibration X' which has the properties of case 2 of subsect. 2.4. With Lemma 2.2 F can be extended to a topological map $\overline{F} : \overline{X} \to \overline{X'}$ which is real analytic in all points of ∂X except the edges. We now define:

Definition 3.15 $\rho(\overline{X}, s, z) := \rho(\overline{X}', F(s), z)$ *and* $r_s(z) := r_{F(s)}(z)$

We have to prove that this is well defined. If z–locally X is biholomorphically equivalent with two smooth fibrations, e.g. $F' : X \to X'$ and $F'' : X \to X''$ then $F'' \circ (F')^{-1} : X' \to X''$ is a biholomorphic fiber preserving mapping which z–locally can be extended biholomorphically into a neighborhood of $\overline{X'}$ by Lemma 2.1. Now the assertion follows with Prop. 3.2 and Prop. 3.8. □

Remark: $\rho(\overline{X}, s, z)$ and $r_s(z)$ are defined in such a way that all results of the subsects. 3.1 and 3.2 hold for X if we introduce coordinates in X in such a way that ∂X becomes real analytically smooth.

Under some additional assumptions on the edges of X we can get $r_s(z)$ without using F and X':
In a smooth border point (w_0, z_0) of ∂X , let $\xi_{z_0}(w_0)$ be the vector of the complex tangent at ∂X with $\pi_*(\xi_{z_0}(w_0)) = \frac{\partial}{\partial z}$. It is uniquely determined with these properties. In the same way as in subsect. 3.2, we can introduce the standart vector field γ_s in $\overline{X} \setminus \{edges\}$, we only have to use Lemma 2.2 instead of Lemma 2.1.
Then we have:

Proposition 3.16 *Assume that for every edge point* $P = (w_0, 0)$ *of* ∂X_0 *there exists a neighborhood* $V(P)$ *in* $\mathbb{C} \times \{0\}$ *s.th. after a holomorphic coordinate transformation* T *in* $V(P)$ $\partial X_0 \cap V(P)$ *consists of two straight lines which intersect in* P *under an angle of less than* π. *Then for every* $z \in D$ *there exists a unique continuous function* $f : \overline{X_z} \to \mathbb{C}$ *which is harmonic in* X_z *and for which we have* $(\xi_z - \frac{\partial}{\partial z}) = f \cdot \gamma_s$ *on* $\partial X_z \setminus \{edges\}$. *For this function we have* $r_s(z) = -i \int_{X_z} \overline{\partial} f \wedge \partial \overline{f}$.

Remarks: 1) Since ∂X is a cartesian product in the $(\tilde{w}, z)$–coordinates the assumptions of the Lemma are fulfilled for X_0 iff they are fulfilled for any other X_z, $z \in D$. So the special choice of X_0 is unimportant.
2) The assertion of the Lemma is false if there occurs any angle greater than π.
3) The assumption of the Lemma can be weakened (c.f. [P], chapter 10, e.g. p.308). Nevertheless it seems doubtful if the assertion is still true if we only assume that all angles which occur are less than π (c.f. [P], [C,p.94]).
Proof: 1) To simplify writing we assume $z = 0$. Let F and X' with coordinates (w', z) be like at the beginning of this subsection with $X'_0 = \Delta$. Let $G = F^{-1}$ and E be the set of the edges of $\overline{X}$. We may assume that in X and in X' s is the zero section. Let further ξ_0 (for X_0) resp. ξ'_0 (for X'_0) be like above resp. like in subsect. 3.1. We define $\eta = \xi_0 - \frac{\partial}{\partial z}$, $\eta' = \xi'_0 - \frac{\partial}{\partial z'}$. By Lemma 3.6 there exists a continuous function $f' : \overline{X'_0} \to \mathbb{C}$ which is harmonic in X_0 and for which we have $\eta' = f' \cdot \gamma_s$ on $\partial X'_0$. With the same argument as in the proof of Prop. 3.8 (applied to $\overline{X} \setminus E$) we get a continuous function $h : (\overline{X_0} \setminus E) \to \mathbb{C}$ which is holomorphic in X_0 and fulfills

$$G_*(\frac{\partial}{\partial z'}) - \frac{\partial}{\partial z} = h \cdot \gamma_s \;\; in\; \overline{X_0} \setminus E \;,\;\; \eta = (f' + h) \cdot \gamma_s \;\; on\; \partial X_0 \setminus E \qquad (18)$$

2) We wish to prove that h can be extended continuously into $\overline{X_0}$.
Let F_1 be the canonic biholomorphic map which maps X (with the coordinates (w, z)) to the manifold $X^1 = X$ (with the both coordinates (w, z) and $(\tilde{w}, z)$). Let F_2 be the canonic biholomorphic map which maps X^1 to the manifold $X^2 = X$ (with the both coordinate systems (w, z) and (w^*, z)), where the w^*–coordinate is defined like in the proof of Lemma 2.2, especially we have $\partial X = \partial\Delta \times D$ in these coordinates. If we define $F_3 = F \circ F_1^{-1} \circ F_2^{-1}$ and $G_i = F_i^{-1}$, we have factorisations $F = F_3 \circ F_2 \circ F_1$ on X and $G = G_1 \circ G_2 \circ G_3$ on X'. We know that G_1 is still defined in a neighborhood of $\overline{X^1}$, and with Lemma 2.1 we show like in the proof of Lemma 2.2 that G_3 can 0–locally be extended into a neighborhood of $\overline{X'}$. We further know from the construction of the coordinate function $w^*(\tilde{w})$ in the proof of Lemma 2.2 that G_2 can 0–locally be extended over all border points except the edges.
Let now P be an edge point of $\overline{X_0}$ and $V(P)$ a small neighborhood in $\mathbb{C} \times \{0\}$. If

$$\begin{pmatrix} g_{i1} & g_{i2} \\ 0 & 1 \end{pmatrix}$$

are the functional matrices of $G_1 : X^1(in(\tilde{w}, z)) \to X(in(w, z))$ resp. of $G_2 : X^2(in(w^*, z)) \to X^1\ (in\ (\tilde{w}, z))$ resp. of $G_3 : X'\ (in\ (w', z)) \to X^2\ (in\ (w^*, z))$, we know that g_{11}, g_{12} resp. g_{31}, g_{32} still are defined in a neighborhood of $\overline{X^1}$ resp. of $\overline{X'}$ and that g_{11} and g_{31} have no zeros since G_1 and G_3 are biholomorphic. Further we know that $g_{22} \equiv 0$ because w^* only is a function of $\tilde{w}$ and that g_{21} can holomorphically be extended over all border points except the edges. Now we have in $(V(P) \cap \overline{X_0}) \setminus \{P\}$:

$$G_*(\frac{\partial}{\partial z'}) - \frac{\partial}{\partial z} = (g_{11}g_{21}g_{32} + g_{12})\frac{\partial}{\partial w}$$

$$\gamma_s\ (in\ \overline{X_0} \setminus E)\ = G_*(w'\frac{\partial}{\partial w'}) = (g_{11}g_{21}g_{31}w')\frac{\partial}{\partial w}$$

$$h = \frac{g_{32}}{g_{31}w'} + \frac{g_{12}}{g_{11}g_{31}w'} \cdot \frac{1}{g_{21}}$$

Since $\frac{g_{32}}{g_{31}w'}$ and $\frac{g_{12}}{g_{11}g_{31}w'}$ are still continuous in P, all depends on $(g_{21})^{-1}$, but this is nothing but the derivative of the function $w^*(\tilde{w})$ defined on $F_1(V) \cap X^1$ in $\tilde{w}$–direction. Since the coordinate transformation T of the hypothesis is defined in a neighborhood of P, we may assume that $\partial X^1 \cap F_1(V)$ already consists of two straight lines which intersect in $F_1(P)$ under an angle $\alpha < \pi$, otherwise we would only have to multiply $w^*_{\tilde{w}}(\tilde{w})$ with an in a neighborhood of $F_1(P)$ holomorphic function without zeros, which wouldn't change anything. If $F_1(P) = (\tilde{w}_0, 0)$ we now can make the transformation

$$w' = (\tilde{w} - \tilde{w}_0)^{\frac{\pi}{\alpha}}$$

In the w'–coordinate $\partial X^1 \cap F_1(V)$ is a straight line through $F_1(P)$. We now have a factorisation $w^*(\tilde{w}) = w^*(w'(\tilde{w}))$ on $X^1 \cap F_1(V)$. Since $\pi/\alpha > 1$ the derivative of w' in $\tilde{w}$–direction can be continuously extended into $\partial X^1 \cap F_1(V)$. The function $w^*(w')$ can be extended holomorphically into $F_1(P)$ by the Schwarz reflection principle, hence the derivative of w^* in w'–direction can be extended holomorphically into the image of $\underline{\partial X^1 \cap F_1(V)}$. So the derivative of $w^*(\tilde{w})$ in $\tilde{w}$–direction can be extended continuously into $F_1(V) \cap \overline{X^1}$, which completes the proof of 2).
3) Define $f = f' + h$. Then f is continuous in $\overline{X_0}$, harmonic in X_0 and by (18) the equation $\eta = f \cdot \gamma_s$ on $\partial X_0 \setminus E$ holds. If f_1 is an other function with these properties, we have $f - f_1 = 0$ on $\partial X_0 \setminus E$, hence by the continuity we have this equation on ∂X_0 and by the harmonicity we have it on $\overline{X_0}$. So f is uniquely determined by those properties. Since h is holomorphic we have $\overline{\partial} f = \overline{\partial} f'$, hence $r_s(0) = -i \int_{X_0} \overline{\partial} f \wedge \partial \overline{f}$, so everything is proved. □

4. Assume first that X is of case 2 of subsect. 2.4 and let $z \in D$ be arbitrary. Then we can compute the real number $r_s(z)$ as follows:

step 1 By the classical Riemann mapping theorem we can get a biholomorphic mapping $f : X_z \to \Delta$ with $f(s(z)) = 0$. This gives us the vector field γ_s on $\overline{X_z}$.

step 2 Next we have to compute the field $\xi_z - \frac{\partial}{\partial z}$ on ∂X_z.

step 3 Since $\gamma_s \neq 0$ on ∂X_z (because f can be extended biholomorphically into a neighborhood of $\overline{X_z}$ in $\mathbb{C}$, c.f. Lemma 2.1), we can 'devide' $\xi_z - \frac{\partial}{\partial z}$ through γ_s and get a real analytic border function b on ∂X_z. Then we solve the classical Dirichlet problem for $\overline{X_z}$ with the border values b. At last we make the $\overline{\partial}$–derivation and integrate the result over X_z.

For the steps 1 and 3 there exist a lot of algorithms. So we only have to deal with the problem how to compute the field $\xi_z - \frac{\partial}{\partial z}$ on ∂X_z. Denote $w = x_1 + ix_2$, $z = x_3 + ix_4$ and $x = (x_1, ..., x_4)$. Let for $x \in \partial X$ $(a_1(x), ..., a_4(x))$ be a vector standing vertical on ∂X in x. We always can compute such a vector, e.g. if ∂X is locally around x given by $\{\phi = 0\}$ with a real analytic function $\phi(x)$, than we have $d\phi(x) = \sum_{i=1}^4 a_i(x) dx_i$. From the assumption that $\pi : \partial X \to D$ is regular it follows that $a_1(x) \neq 0$ or $a_2(x) \neq 0$. Then the complex tangent in x is spanned over $\mathbb{R}$ by the two vectors

$$\omega_1 = v(x)\frac{\partial}{\partial x_1} + u(x)\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_4}, \; \omega_2 = -u(x)\frac{\partial}{\partial x_1} + v(x)\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}$$

$$\textit{with } u(x) = \frac{a_1(x)a_3(x) + a_2(x)a_4(x)}{a_1^2(x) + a_2^2(x)}, \; v(x) = \frac{a_1(x)a_4(x) - a_2(x)a_3(x)}{a_1^2(x) + a_2^2(x)}$$

and we have

$$\xi_z(w) = \omega_2(x)$$

Hence written holomorphically we have

$$\xi_z(w) - \frac{\partial}{\partial z} = (-u(x) + i \cdot v(x))\frac{\partial}{\partial w}$$

Assume now that X is of case 1 of subsect. 2.4 and that the assumptions of Prop. 3.16 are fulfilled. Then we can compute $r_s(z)$ as follows: "step 1" and "step 2"are the same as for fibrations X without edges (in step 1 we get, of course, γ_s only in $\overline{X_z} \setminus \{edges\}$). In "step 3", we first have to show that γ_s has no zeros on $\partial X_z \setminus \{edges\}$. This follows since we can extend the mapping $f : X_z \to \Delta$ over those border points by the Schwarz reflection principle and since f is biholomorphic and ∂X_z and $\partial\Delta$ are real analytically smooth around those border points. We get our border function only on $\partial X_z \setminus \{edges\}$, but we know from Prop. 3.16 that we can extend it to a continuous function on ∂X_z. Now we can continue as for fibrations without edges.
The field $\xi_z - \frac{\partial}{\partial z}$ can be computed very easily since we have the coordinates $(\tilde{w}, z)$ z–locally in a neighborhood of ∂X in which ∂X is a cartesian product, hence $\xi_z = \frac{\partial}{\partial z}$ in these coordinates. By the coordinate transformation from $(\tilde{w}, z)$ to (w, z) we get ξ_z in (w, z)–coordinates.

As an example we compute the field $\xi_z(w)$ on ∂X for

$$X = \{(w, z) \in \mathbb{C} \times \Delta : \; w\overline{w} + z\overline{z} < 1\}\, , \; s = 0 \times \Delta$$

We have $\phi(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 1$, $a(x) = 2(x_1, x_2, x_3, x_4)$ and hence

$$\xi_z(w) - \frac{\partial}{\partial z} = \frac{-(x_1x_3 - x_2x_4) + i(x_1x_4 - x_2x_3)}{x_1^2 + x_2^2}\frac{\partial}{\partial w} = \frac{-w\overline{z}}{w\overline{w}}\frac{\partial}{\partial w} = -\frac{\overline{z}}{\overline{w}}\frac{\partial}{\partial w}$$

This proves the formula (15) which we already have used in the proof of Theorem 3.11.

4 Infinitesimal Deformation and Gauss Curvature

1. Let X be of case 1 or of case 2 of subsect. 2.4. Let $\chi(z)$ be a holomorphic field of vectors on $s(D)$ in direction of the fibers. If $ds^2 = g(w, z)dwd\overline{w}$ denotes the hyperbolic metric on the fibers $X_z, z \in D$, we have shown in Lemma 2.5 that $g(w, z)$ is a real analytic function. We now can evaluate the field $\chi(z)$ by this metric and get a real analytic function

$$h(z) = [g(s(z), z)dwd\overline{w}](\chi(z)) = g(s(z), z)|\chi(z)|^2$$

where $|\chi(z)|$ denotes the euclidean metric in the w-coordinate.
By the way of its definition $h(z)$ is independant of z–locally holomorphic coordinate transformations of the form

$$T : X \to X\, ; \; (w, z) \to (t(w, z), z)\, , \; t : X \to \mathbb{C} \textit{ holomorphic} \tag{19}$$

It also can be computed easily that if we make a holomorphic coordinate transformation in D then $h(z)$ transforms like a function.
For a *fixed coordinate z in D* we can define a metric $ds^2 = h(z)dzd\overline{z}$ with Gauss curvature $\mathcal{K}$. By introducing special coordinates we can compute $\mathcal{K}$:

Theorem 4.1 *Assume that z–locally we have coordinate transformations T_1 and T_2 of the form (19) s.th. in the T_1–coordinates X has the properties of case 2 of subsect. 2.4 and ∂X is pseudoconvex and in the T_2–coordinates we have $X_z = \Delta$ and $s(z) = 0$. Then we have*

$$\mathcal{K}(z) = \frac{-\triangle_z \log h(z)}{2h(z)} = \frac{-(2b(z) + \frac{4}{\pi} r_s(z))}{8|\chi(z)|^2}$$

where $b(z) \geq 0$ and $r_s(z)$ are computed in the $\underline{T_1\text{–coordinates}}$ and $b(z) > 0$ iff in those coordinates ∂X is strongly pseudoconvex in some point of ∂X_z. $|\chi(z_0)|$ denotes the euclidean norm computed in the $\underline{T_2\text{–coordinates}}$.

Theorem 4.1 will be proved in the subsects. 4.2 to 4.4.
We always can find T_2–coordinates (c.f. Lemma 2.6). If X is of case 2 of subsect. 2.4 and ∂X is pseudoconvex then we don't need any T_1–coordinates, in case 1 we can find some:

Proposition 4.2 *Let X have the properties of case 1 of subsect. 2.4. Then (with the notations of Theorem 4.1) we can find T_1–coordinates in which no point of ∂X is strictly pseudoconvex, hence $b(z) \equiv 0$ (by the Theorem).*

Proof: From Lemma 2.6 we know that z–locally there exists a topological fiber preserving and in the interior biholomorphical map $F : \overline{X} \to \overline{X'}$ onto a fibration X' which is of case 2 of subsect. 2.4. By assumption there z–locally exists a neighborhood $U(\partial X) \subset \mathbb{C} \times D$ and holomorphic coordinates $(\tilde{w}, z)$ s.th. in these coordinates we have $\partial X = \partial X_z \times D$. In the same way as in the proof of Lemma 2.2 we can introduce a coordinate (w^*, z) in $U \cap X$ s.th. in these coordinates we have $\partial X = \partial \Delta \times D$. This gives a biholomorphic map $\tilde{F} : U \cap X \; (in(w^*, z)) \to X' \setminus F(X \setminus U)$, for which in both sets the exterior border is real analytically smooth. Hence an application of Lemma 2.1 yields that z–locally there exists a fiber preserving biholomorphical extension over the exterior borders. Since $\partial \Delta \times D$ is pseudoconvex and not strictly pseudoconvex in any point this follows for $\partial X'$. □

2. The first equation in Theorem 4.1 is a standart formula for the Gauss curvature. Since the numerator and the denominator of the second term of this equation are independant of coordinate transformations of the type (19) we may compute them in different coordinates. Assume first that X has coordinates s.th. $X_z = \Delta$ and $s(z) = 0$. The hyperbolic metric on Δ is $\frac{4}{(1-w\overline{w})^2} dw d\overline{w}$, hence $h(z) = 4 dw d\overline{w}(\chi(z)) = 4|\chi(z)|^2$.
The computation of the numerator is much more difficult. We only do this for $z = 0$ since then notations become easier. Assume for the rest of this section that X has the properties of case 2 of subsect. 2.4 and ∂X is pseudoconvex. By Lemma 2.6 and Lemma 2.1 there 0–locally exists a biholomorphic fiber preserving coordinate transformation defined in a neighborhood of $\overline{X}$ such that in the new coordinates we have $X|\check{D} = \Delta \times \check{D}$ and $s = 0 \times D$. We may compute the numerator in the new coordinates —we only have to show at the end that the results of our computation are independant of such coordinate transformations.
First we have:

$$-\triangle_z \log h(z) = -\triangle_z[\log g(0, z) + \log \chi(z)^w + \log \overline{\chi}(z)^w] = -\triangle_z \log g(0, z) \tag{20}$$

where $\chi(z)^w$ resp. $g(w, z)$ denote the coefficient of the vector $\chi(z)$ of $\partial/\partial w$ resp. the coefficient of the hyperbolic metric on X_z in the coordinate w on ∂X_z. The second equality holds since $\chi(z)$ is a holomorphic vector field.
The hyperbolic metric on the fibers of X was constructed by a fiberwise pulling back of the hyperbolic metric on Δ to X by the map $F : X \to \Delta \times D$ which we constructed in Lemma 2.4. So we have:

$$g(0, z) = 4|F_w(0, z)|^2 \tag{21}$$

It was $F(w,z) = w \cdot e^{(q+it)(w,z)}$, where $q(w,z)$ was a continuous, in the interior real analytic and on the fibers harmonic extension from the function $-\log|w|$ defined on ∂X to $\overline{X}$. So we have $|F_w(0,z)| = e^{q(o,z)}$. We set $z = x + iy$. Since we have $X_{iy} \equiv \Delta$, we have $q(w,iy) \equiv 0$ and hence $\frac{\partial^2}{\partial y^2} q(0,iy) \equiv 0$. So it follows with (20) and (21):

$$-\triangle_z \log h(0) = -\frac{\partial^2}{\partial x^2} 2q(0,x)|_{x=0} \tag{22}$$

3. In this subsection we want to compute $\frac{\partial^2}{\partial x^2} q(0,x)|_{x=0}$.
In subsect. 1.2 it was shown that 0–locally we have

$$\partial X = \{(w,z) : \log|w| = h(\theta,z)\}$$

with $\theta = arc(w)$ and h being periodic in θ (with period 2π) and real analytic in θ and z. Moreover we have $h(\theta,iy) \equiv 1$. We put $w^* := \log w =: \tau + i\theta$, $\alpha(\theta,y) := (\log h(\theta,iy))_x$, $\beta(\theta,y) := (\log h(\theta,iy))_{xx}$ and consider (τ,θ,x,y) as real coordinates in $\mathbb{C}^* \times D$. Then $(\alpha(\theta,y),0,1,0)$ is a tangent vector at ∂X over $\check{D}$, the multiplication with $\sqrt{-1}$ gives $(0,\alpha(\theta,y),0,1)$, and this is a tangent vector again. So the complex tangent is spanned over $\mathbb{R}$ by these two vectors.
The boundary of X is given by $\{\phi = 0\}$ with $\phi = \tau - \log h(\theta, x+iy)$. If $L(\phi)$ denotes the Levi form we get after some computation:

$$4L(\phi) = -\beta(dx^2 + dy^2) + 2\alpha_\theta(dyd\tau - dxd\theta)$$

If we evaluate $L(\phi)$ on the complex tangent spanned by our both complex tangent vectors above we get, since we have assumed that ∂X is pseudoconvex:

$$-\beta(\theta,y) \geq 0 \tag{23}$$

and $-\beta(\theta,y) > 0$ iff ∂X is strictly pseudoconvex in $(\log(h(\theta,iy)),\theta,0,y) \in \partial X$.
To get equations more elegant we introduce the following notation: If var is any variable and at the end of an equation $(mod\ var^n)$ occurs we mean that the equation holds up to an in x and w or θ real analytic function multiplied with var^n. Further we denote $\alpha(\theta) := \alpha(\theta,0)$, $\beta(\theta) := \beta(\theta,0)$. If we now devellop $\log h(\theta,x)$ in a power series in x and $\alpha(\theta)$, $\beta(\theta)$ in Fourier series in θ, we get:

$$\log h(\theta,x) = x\alpha(\theta) + \frac{1}{2}x^2\beta(\theta)\ (mod\ x^3) \tag{24}$$

$$\alpha(\theta) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}\ ,\ \beta(\theta) = \sum_{n\in\mathbb{Z}} b_n e^{in\theta}$$

with $a_{-n} = \overline{a_n}$, $b_{-n} = \overline{b_n}$. We extend $\alpha(\theta)$, $\beta(\theta)$ harmonically from $\partial\Delta$ into Δ and get:

$$f_1(w) := a_0 + \sum_{n\in\mathbb{N}} (a_n w^n + \overline{a_n}\,\overline{w}^n)\ ,\ f_2(w) := b_0 + \sum_{n\in\mathbb{N}} (b_n w^n + \overline{b_n}\overline{w}^n)$$

Since $\alpha(\theta)$, $\beta(\theta)$ are real analytic in θ, we know from the Proof of Lemma 3.6 that the series $f_1(w)$, $f_2(w)$ have a convergence radius greater than one. So especially the function $(f_1)_\tau(w)$ exists and has the fourier series $\sum_{n\in\mathbb{Z}} |n| a_n e^{in\theta}$ on $\partial\Delta$. Hence $(f_1)_\tau(w)\alpha(\theta)$ has the Fourier series

$$\sum_{n\in\mathbb{Z}} \Big(\sum_{m+l=n} |m| a_m a_l\Big) e^{in\theta} = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}$$

on $\partial\Delta$ with $c_{-n} = \overline{c_n}$ and the harmonic continuation

$$f_3(w) := c_0 + \sum_{n\in\mathbb{N}} (c_n w^n + \overline{c_n}\,\overline{w}^n)$$

from $\partial\Delta$ to Δ which again has a convergence radius greater than one. Lastly we denote

$$e(w,x) := x\, f_1(w) + \frac{1}{2}x^2 f_2(w) - x^2 f_3(w)$$

In our logarithmical coordinates we have on ∂X with (24):

$$\log|w| - e(w,x) = x\,\alpha(\theta) + \frac{1}{2}x^2\beta(\theta) - x\, f_1(e^{\tau+i\theta}) - \frac{1}{2}x^2 f_2(e^{\tau+i\theta}) + x^2 f_3(e^{\tau+i\theta})\,(mod\, x^3) \quad (25)$$

We have

$$f_1(e^{\tau+i\theta}) = \alpha(\theta) + \tau (f_1)_\tau(e^{i\theta})\,(mod\,\tau^2)$$
$$f_2(e^{\tau+i\theta}) = \beta(\theta)\,(mod\,\tau)\,,\ f_3(e^{\tau+i\theta}) = (f_1)_\tau(e^{i\theta})\alpha(\theta)\,(mod\,\tau)$$

With (24) we have $\tau = x\,\alpha(\theta) + \frac{1}{2}x^2\beta(\theta)\,(mod\,x^3)$ on ∂X. If we put this and the three Taylor expansions for the f_i into (25), we get $\log|w| - e(w,z) = 0\,(mod\,x^3)$ on ∂X, so we have proved:

$$\log|w| = e(w,x) + x^3\gamma(w,x)\ on\ \partial X \quad (26)$$

with real analytic functions $e(w,x)$ and $\gamma(w,x)$.
With the same argument as used for the proof of Lemma 3.6 we can solve the Dirichlet problems for $e(w,x)$ and $\gamma(w,x)$. So we get: $-q(w,x) = q_1(w,x) + x^3 q_2(w,x)$, where the q_i are continuous on $\overline{X|_{\hat{D}}}$ (where $\hat{D} := \{z \in D :\ y = 0\}$), real analytic and on the fibers harmonic in $X|_{\hat{D}}$. On $\partial X|_{\hat{D}}$ we have $-q = \log|w|$, $q_1 = e(w,x)|_{\partial X_{\hat{D}}}$, $q_2 = \gamma(w,x)$. Since $q_1(w,x)$ is uniquely determined with these properties we have $q_1(w,x) = e(w,x)$. Hence we have with (22):

$$-\triangle_z \log h(0) = -2q(0,0)_{xx} = 2f_2(0,0) - 4f_3(0,0) = 2b_0 - 4\sum_{m+l=0}|m|a_m a_l =$$

$$= 2b_0 - 8\sum_{n\in IN} n\cdot|a_n|^2 \quad (27)$$

What remains to do now is to 'interprete' these two summands and to show then that they are invariant under fiber preserving holomorphic coordinate transformations defined in a neighborhood of $\overline{X}$.
We have $b_0 = \frac{1}{2\pi}\int_0^{2\pi}\beta(\theta)d\theta$. So with (23) we have $-b_0 \geq 0$ and $-b_0 > 0$ iff ∂X is strictly pseudoconvex in at least one point of ∂X_0.

4. We wish to prove:

$$\sum_{n\in IN} n\cdot|a_n|^2 = \frac{1}{2\pi}\cdot r_s(0)$$

We know that in coordinates (τ,θ,x,y) of subsect. 4.2 the vector $\alpha(\theta)\partial/\partial\tau + \partial/\partial x$ is in the complex tangent. If we transform back to the coordinates (w,z) this vector becomes $\alpha(\theta)\cdot w\cdot\frac{\partial}{\partial w} + \frac{\partial}{\partial z}$ and still lies on the complex tangent. So we have

$$\xi_0(w) = \alpha(\theta)\cdot w\cdot\frac{\partial}{\partial w} + \frac{\partial}{\partial z}\ on\ \partial X_0$$

Since $X_0 = \Delta$ we have $\gamma_s = w\frac{\partial}{\partial w}$ and by subsect. 4.2 $f_1(w) = a_0 + \sum_{n\in IN}(a_n w^n + \overline{a_n}\,\overline{w}^n)$ gives the harmonic continuation for $\alpha(\theta)$. We set $w = u+iv$ and compute:

$$r_s(0) = -i\int_\Delta \overline{\partial} f_1 \wedge \partial\overline{f_1} = -i\int_\Delta (\sum_{n\in IN}(n\overline{a_n}\,\overline{w}^{n-1})(\sum_{m\in IN} a_m w^{m-1})d\overline{w}\wedge dw =$$

$$= \sum_{m,n\in IN} mn\overline{a_n}a_m \int_\Delta \overline{w}^{n-1}w^{m-1}2du\wedge dv$$

By transformation on polar coordinates we compute from this:

$$r_s(0) = \sum_{n\in IN} n^2|a_n|^2 2\frac{\pi}{n} = 2\pi \sum_{n\in IN} n|a_n|^2$$

So we also have computed the second summand of (27).
We know from Prop. 3.8 that $r_s(0)$ is invariant under fiber preserving biholomorphic coordinate transformations defined 0–locally in a neighborhood of $\overline{X}$. Since the numerator also is independant of those transformations, the number $b(0)$ is, too. Since the property that ∂X is strictly pseudoconvex or not is also kept under these coordinate transformations the proof of Theorem 4.1 is complete. □

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