

RAMIFICATION OF THE GAUSS MAP OF COMPLETE MINIMAL SURFACES IN \mathbb{R}^3 AND \mathbb{R}^4 ON ANNULAR ENDS

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ABSTRACT. In this article, we study the ramification of the Gauss map of complete minimal surfaces in \mathbb{R}^3 and \mathbb{R}^4 on annular ends. We obtain results which are similar to the ones obtained by Fujimoto ([4], [5]) and Ru ([13], [14]) for (the whole) complete minimal surfaces, thus we show that the restriction of the Gauss map to an annular end of such a complete minimal surface cannot have more branching (and in particular not avoid more values) than on the whole complete minimal surface. We thus give an improvement of the results on annular ends of complete minimal surfaces of Kao ([8]).

RESUME. Dans ce travail nous obtenons des théorèmes de ramification de l'application de Gauss de certaines classes de surfaces minimales complètes dans \mathbb{R}^3 et \mathbb{R}^4 .

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1. INTRODUCTION

Let M be a non-flat minimal surface in \mathbb{R}^3 , or more precisely, a non-flat connected oriented minimal surface in \mathbb{R}^3 . By definition, the Gauss map G of M is a map which maps each point $p \in M$ to the unit normal vector $G(p) \in S^2$ of M at p . Instead of G , we study the map $g := \pi \circ G : M \rightarrow \mathbb{P}^1(\mathbb{C})$, where $\pi : S^2 \rightarrow \mathbb{P}^1(\mathbb{C})$ is the stereographic projection. By associating a holomorphic local coordinate $z = u + \sqrt{-1}v$ with each positive isothermal coordinate system (u, v) , M is considered as an open Riemann surface with a conformal metric ds^2 and by the assumption of minimality of M , g is a meromorphic function on M .

In 1988, H. Fujimoto ([4]) proved Nirenberg's conjecture that if M is a complete non-flat minimal surface in \mathbb{R}^3 , then its Gauss map can omit at most 4 points, and the bound is sharp. After that, he also extended that result for minimal surfaces in \mathbb{R}^m .

In 1993, M. Ru ([14]) refined these results by studying the Gauss maps of minimal surfaces in \mathbb{R}^m with ramification. But for our purpose, we here only introduce the case $m = 3$. To give that result, we recall some definitions.

One says that $g : M \rightarrow \mathbb{P}^1(\mathbb{C})$ is *ramified over a point* $a = (a_0 : a_1) \in \mathbb{P}^1(\mathbb{C})$ with multiplicity at least e if all the zeros of the function $a_0g_1 - a_1g_0$ have orders at least e , where $g = (g_0 : g_1)$ is a reduced representation. If the image of g omits a , one will say that g is *ramified over a with multiplicity* ∞ . Ru proved :

Theorem A. *Let M be a non-flat complete minimal surface in \mathbb{R}^3 . If there are q ($q > 4$) distinct points $a^1, \dots, a^q \in \mathbb{P}^1(\mathbb{C})$ such that the Gauss map of M is ramified over a^j with multiplicity at least m_j for each j , then $\sum_{j=1}^q (1 - \frac{1}{m_j}) \leq 4$.*

To prove this result, he constructed a pseudo-metric with negative curvature with ramification and used the previous argument of Fujimoto.

On the other hand, in 1991, S. J. Kao ([8]) used the ideas of Fujimoto ([4]) to show that the Gauss map of an end of a non-flat complete

minimal surface in \mathbb{R}^3 that is conformally an annulus $\{z : 0 < 1/r < |z| < r\}$ must also assume every value, with at most 4 exceptions. In 2007, L. Jin and M. Ru ([7]) extended Kao's result to minimal surfaces in \mathbb{R}^m . Kao ([8]) proved :

Theorem B. *The Gauss map g on an annular end of a non-flat complete minimal surface in \mathbb{R}^3 assumes every value on the unit sphere infinitely often, with the possible exception of four values.*

A natural question is whether a result as in Theorem A for the ramification of the Gauss map still holds on an annular end of a non-flat complete minimal surface in \mathbb{R}^3 . In this paper we give an affirmative answer :

Theorem 1. *Let M be a non-flat complete minimal surface in \mathbb{R}^3 and let $A \subset M$ be an annular end of M which is conformal to $\{z : 0 < 1/r < |z| < r\}$, where z is a conformal coordinate. If there are q ($q > 4$) distinct points $a^1, \dots, a^q \in \mathbb{P}^1(\mathbb{C})$ such that the restriction of the Gauss map of M to A is ramified over a^j with multiplicity at least m_j for each j , then*

$$\sum_{j=1}^q \left(1 - \frac{1}{m_j}\right) \leq 4. \quad (1.1)$$

Moreover, (1.1) still holds if we replace, for all $j = 1, \dots, q$, m_j by the limit inferior of the orders of the zeros of the function $a_0^j g_1 - a_1^j g_0$ on A (where $g = (g_0 : g_1)$ is a reduced representation and $a^j = (a_0^j : a_1^j)$) and in particular by ∞ if g takes the value a^j only a finite number of times on A .

Theorem 1 gives in particular the following generalization of Theorem B of Kao :

Corollary 2. *If the Gauss map g on an annular end of a non-flat complete minimal surface in \mathbb{R}^3 assumes four values on the unit sphere only finitely often, it takes any other value infinitely often without ramification.*

Moreover, we also would like to consider the Gauss map of complete minimal surfaces M immersed in \mathbb{R}^4 , this case has been investigated by various authors (see, for example Osserman ([11]), Chen ([2]), Fujimoto ([5]) and Kawakami ([9])). In this case, the Gauss map of M may be identified with a pair of meromorphic functions $g = (g^1, g^2)$ which is introduced in §4 (see also Osserman ([11]), Fujimoto ([5]) and Kawakami ([9])). We shall prove the following result which again shows that the restriction of the Gauss map to an annular end cannot have more branching (and in particular not avoid more values) than on the whole complete minimal surface immersed in \mathbb{R}^4 :

Theorem 3. *Suppose that M is a complete non-flat minimal surface in \mathbb{R}^4 and $g = (g^1, g^2)$ is the Gauss map of M . Let A be an annular end of M which is conformal to $\{z : 0 < 1/r < |z| < r\}$, where z is a conformal coordinate. Let a^{11}, \dots, a^{1q_1} respectively a^{21}, \dots, a^{2q_2} be q_1 ($q_1 > 2$) respectively q_2 ($q_2 > 2$) distinct points in $\mathbb{P}^1(\mathbb{C})$.*

(i) *In the case $g^l \not\equiv \text{constant}$ ($l = 1, 2$), if g^l is ramified over a^{lj} with multiplicity at least m_{lj} for each j ($l = 1, 2$) on A , then $\gamma_1 = \sum_{j=1}^{q_1} (1 - \frac{1}{m_{1j}}) \leq 2$, or $\gamma_2 = \sum_{j=1}^{q_2} (1 - \frac{1}{m_{2j}}) \leq 2$, or*

$$\frac{1}{\gamma_1 - 2} + \frac{1}{\gamma_2 - 2} \geq 1.$$

(ii) *In the case where g^1 or g^2 is constant, say $g^2 \equiv \text{constant}$, if g^1 is ramified over a^{1j} with multiplicity at least m_{1j} for each j , we have the following :*

$$\gamma_1 = \sum_{j=1}^{q_1} (1 - \frac{1}{m_{1j}}) \leq 3.$$

Moreover, the results still hold if we replace, for all a^{lj} ($j = 1, \dots, q_l$; $l = 1, 2$) the m_{lj} by the limit inferior of the orders of the zeros of the function $a_0^{lj} g_1^l - a_1^{lj} g_0^l$ on A (where $g^l = (g_0^l : g_1^l)$ are reduced representations and $a^{lj} = (a_0^{lj} : a_1^{lj})$) and in particular by ∞ if g^l takes the value a^{lj} only a finite number of times on A .

The main idea to prove the theorems is to construct a pseudo-metric with negative curvature with ramification on an annular end, which is

a refinement of the ideas in Ru ([14]). After that we use arguments similar to those used by Kao ([8]) and by Fujimoto ([4], [6]) to finish the proofs.

2. AUXILIARY LEMMAS

Let f be a nonconstant holomorphic map of a disk $\Delta_R := \{z \in \mathbb{C} : |z| < R\}$ into $\mathbb{P}^1(\mathbb{C})$, where $0 < R < \infty$. Take a reduced representation $f = (f_0 : f_1)$ on Δ_R and define

$$\|f\| := (|f_0|^2 + |f_1|^2)^{1/2}, W(f_0, f_1) = W_z(f_0, f_1) := f_0 f_1' - f_1 f_0',$$

where the derivatives are taken with respect to the variable z . Let a^j ($1 \leq j \leq q$) be q distinct points in $\mathbb{P}^1(\mathbb{C})$. We may assume $a^j = (a_0^j : a_1^j)$ with $|a_0^j|^2 + |a_1^j|^2 = 1$ ($1 \leq j \leq q$), and set

$$F_j := a_0^j f_1 - a_1^j f_0 \quad (1 \leq j \leq q).$$

Proposition 4. ([6, Proposition 2.1.6 and 2.1.7]).

- a) If ξ is another local coordinate, then $W_\xi(f_0, f_1) = W_z(f_0, f_1) \cdot \left(\frac{dz}{d\xi}\right)$.
- b) $W(f_0, f_1) \neq 0$ (iff f is nonconstant).

Proposition 5. ([5, Proposition 2.1]). For each $\epsilon > 0$ there exist positive constants C_1 and μ depending only on a^1, \dots, a^q and on ϵ respectively such that

$$\Delta \log \left(\frac{\|f\|^\epsilon}{\prod_{j=1}^q \log(\mu \|f\|^2 / |F_j|^2)} \right) \geq \frac{C_1 \|f\|^{2q-4} |W(f_0, f_1)|^2}{\prod_{j=1}^q |F_j|^2 \log^2(\mu \|f\|^2 / |F_j|^2)}$$

Lemma 6. Suppose that $q - 2 - \sum_{j=1}^q \frac{1}{m_j} > 0$ and that f is ramified over a^j with multiplicity at least m_j for each j ($1 \leq j \leq q$). Then there exist positive constants C and $\mu (> 1)$ depending only on a^j and m_j ($1 \leq j \leq q$) which satisfy the following : If we set

$$v := \frac{C \|f\|^{q-2-\sum_{j=1}^q \frac{1}{m_j}} |W(f_0, f_1)|}{\prod_{j=1}^q |F_j|^{1-\frac{1}{m_j}} \log(\mu \|f\|^2 / |F_j|^2)}$$

on $\Delta_R \setminus \{F_1 \cdots F_q = 0\}$ and $v = 0$ on $\Delta_R \cap \{F_1 \cdots F_q = 0\}$, then v is continuous on Δ_R and satisfies the condition

$$\Delta \log v \geq v^2$$

in the sense of distribution.

Proof. First, we prove the continuousness of v .

Obviously, v is continuous on $\Delta_R \setminus \{F_1 \cdot \dots \cdot F_q = 0\}$.

Take a point z_0 with $F_i(z_0) = 0$ for some i . Then $F_j(z_0) \neq 0$ for all $j \neq i$ and $\nu_{F_i}(z_0) \geq m_i$. Changing indices if necessary, we may assume that $f_0(z_0) \neq 0$, then $a_0^i \neq 0$. Hence, we get

$$\nu_W(z_0) = \nu_{\frac{(a_0^i \frac{f_1}{f_0} - a_1^i)'}{a_0^i}}(z_0) = \nu_{\frac{(F_i/f_0)'}{a_0^i}}(z_0) = \nu_{F_i}(z_0) - 1.$$

Thus,

$$\begin{aligned} \nu_{v \prod_{j=1}^q \log(\mu \|f\|^2 / |F_j|^2)}(z_0) &= \nu_W(z_0) - \sum_{j=1}^q \left(1 - \frac{1}{m_j}\right) \nu_{F_j}(z_0) \\ &= \nu_{F_i}(z_0) - 1 - \left(1 - \frac{1}{m_i}\right) \nu_{F_i}(z_0) \\ &= \frac{\nu_{F_i}(z_0)}{m_i} - 1 \geq 0. \quad (*) \end{aligned}$$

So, $\lim_{z \rightarrow z_0} v(z) = 0$. This implies that v is continuous on Δ_R .

Now, we choose constants C and μ such that C^2 and μ satisfy the inequality in Proposition 5 for the case $\epsilon = q - 2 - \sum_{j=1}^q \frac{1}{m_j}$. Then we have (by using that $|F_j| \leq \|f\|$ ($1 \leq j \leq q$)) :

$$\begin{aligned} \Delta \log v &\geq \Delta \log \frac{\|f\|^{q-2-\sum_{j=1}^q \frac{1}{m_j}}}{\prod_{j=1}^q \log(\mu \|f\|^2 / |F_j|^2)} \\ &\geq C^2 \frac{\|f\|^{2q-4} |W(f_0, f_1)|^2}{\prod_{j=1}^q |F_j|^2 \log^2(\mu \|f\|^2 / |F_j|^2)} \\ &\geq C^2 \frac{\|f\|^{2q-4-2\sum_{j=1}^q \frac{1}{m_j}} |W(f_0, f_1)|^2}{\prod_{j=1}^q |F_j|^{2-\frac{2}{m_j}} \log^2(\mu \|f\|^2 / |F_j|^2)} \\ &= v^2. \end{aligned}$$

Thus Lemma 6 is proved. \square

Lemma 7. (*Generalized Schwarz's Lemma ([1])*). *Let v be a nonnegative real-valued continuous subharmonic function on Δ_R . If v satisfies*

the inequality $\Delta \log v \geq v^2$ in the sense of distribution, then

$$v(z) \leq \frac{2R}{R^2 - |z|^2}.$$

Lemma 8. *For every δ with $q - 2 - \sum_{j=1}^q \frac{1}{m_j} > q\delta > 0$ and f which is ramified over a^j with multiplicity at least m_j for each j ($1 \leq j \leq q$), there exists a positive constant C_0 such that*

$$\frac{\|f\|^{q-2-\sum_{j=1}^q \frac{1}{m_j}-q\delta} |W(f_0, f_1)|}{\prod_{j=1}^q |F_j|^{1-\frac{1}{m_j}-\delta}} \leq C_0 \frac{2R}{R^2 - |z|^2}.$$

Proof. By using an argument as in (*) of the proof of Lemma 6, the above inequality is correct on $\{F_1 \cdot \dots \cdot F_q = 0\}$ for every $C_0 > 0$ (the left hand side of the above inequality is zero).

If $z \notin \{F_1 \cdot \dots \cdot F_q = 0\}$, using Lemma 6 and Lemma 7, we get

$$\frac{C \|f\|^{q-2-\sum_{j=1}^q \frac{1}{m_j}} |W(f_0, f_1)|}{\prod_{j=1}^q |F_j|^{1-\frac{1}{m_j}} \log(\mu \|f\|^2 / |F_j|^2)} \leq \frac{2R}{R^2 - |z|^2},$$

where C and μ are the constants given in Lemma 6.

On the other hand, for a given $\delta > 0$, it holds that

$$\lim_{x \rightarrow 0} x^\delta \log(\mu/x^2) < +\infty,$$

so we can set

$$\bar{C} := \sup_{0 < x \leq 1} x^\delta \log(\mu/x^2) (< +\infty).$$

Then we have

$$\begin{aligned}
& \frac{\|f\|^{q-2-\sum_{j=1}^q \frac{1}{m_j}-q\delta} |W(f_0, f_1)|}{\prod_{j=1}^q |F_j|^{1-\frac{1}{m_j}-\delta}} \\
&= \frac{\|f\|^{q-2-\sum_{j=1}^q \frac{1}{m_j}} |W(f_0, f_1)|}{\prod_{j=1}^q |F_j|^{1-\frac{1}{m_j}}} \prod_{j=1}^q \left(\frac{|F_j|}{\|f\|} \right)^\delta \\
&= \frac{\|f\|^{q-2-\sum_{j=1}^q \frac{1}{m_j}} |W(f_0, f_1)|}{\prod_{j=1}^q |F_j|^{1-\frac{1}{m_j}} \log(\mu \|f\|^2 / |F_j|^2)} \prod_{j=1}^q \left(\frac{|F_j|}{\|f\|} \right)^\delta \log(\mu \|f\|^2 / |F_j|^2) \\
&\leq \frac{\overline{C}^q \|f\|^{q-2-\sum_{j=1}^q \frac{1}{m_j}} |W(f_0, f_1)|}{\prod_{j=1}^q |F_j|^{1-\frac{1}{m_j}} \log(\mu \|f\|^2 / |F_j|^2)} \\
&\leq \frac{\overline{C}^q}{C} \frac{2R}{R^2 - |z|^2}.
\end{aligned}$$

This proves Lemma 8. \square

We finally will need the following result on completeness of open Riemann surfaces with conformally flat metrics due to Fujimoto :

Lemma 9. ([6, Lemma 1.6.7]). *Let $d\sigma^2$ be a conformal flat metric on an open Riemann surface M . Then for every point $p \in M$, there is a holomorphic and locally biholomorphic map Φ of a disk (possibly with radius ∞) $\Delta_{R_0} := \{w : |w| < R_0\}$ ($0 < R_0 \leq \infty$) onto an open neighborhood of p with $\Phi(0) = p$ such that Φ is a local isometry, namely the pull-back $\Phi^*(d\sigma^2)$ is equal to the standard (flat) metric on Δ_{R_0} , and for some point a_0 with $|a_0| = 1$, the Φ -image of the curve*

$$L_{a_0} : w := a_0 \cdot s \quad (0 \leq s < R_0)$$

is divergent in M (i.e. for any compact set $K \subset M$, there exists an $s_0 < R_0$ such that the Φ -image of the curve $L_{a_0} : w := a_0 \cdot s$ ($s_0 \leq s < R_0$) does not intersect K).

3. THE PROOF OF THEOREM 1

Proof. For convenience of the reader, we first recall some notations on the Gauss map of minimal surfaces in \mathbb{R}^3 . Let $x = (x_1, x_2, x_3) :$

$M \rightarrow \mathbb{R}^3$ be a non-flat complete minimal surface and $g : M \rightarrow \mathbb{P}^1(\mathbb{C})$ its Gauss map. Let z be a local holomorphic coordinate. Set $\phi_i := \partial x_i / \partial z$ ($i = 1, 2, 3$) and $\phi := \phi_1 - \sqrt{-1}\phi_2$. Then, the (classical) Gauss map $g : M \rightarrow \mathbb{P}^1(\mathbb{C})$ is given by

$$g = \frac{\phi_3}{\phi_1 - \sqrt{-1}\phi_2},$$

and the metric on M induced from \mathbb{R}^3 is given by

$$ds^2 = |\phi|^2(1 + |g|^2)^2 |dz|^2 \text{ (see Fujimoto ([6])).}$$

We remark that although the ϕ_i , ($i = 1, 2, 3$) and ϕ depend on z , g and ds^2 do not. Next we take a reduced representation $g = (g_0 : g_1)$ on M and set $\|g\| = (|g_0|^2 + |g_1|^2)^{1/2}$. Then we can rewrite

$$ds^2 = |h|^2 \|g\|^4 |dz|^2, \quad (3.2)$$

where $h := \phi/g_0^2$. In particular, h is a holomorphic map without zeros. We remark that h depends on z , however, the reduced representation $g = (g_0 : g_1)$ is globally defined on M and independent of z . Finally we observe that by the assumption that M is not flat, g is not constant.

Now the proof of Theorem 1 will be given in four steps :

Step 1: We will fix notations on the annular end $A \subset M$. Moreover, by passing to a sub-annular end of $A \subset M$ we simplify the geometry of the theorem.

Let $A \subset M$ be an annular end of M , that is, $A = \{z : 0 < 1/r < |z| < r < \infty\}$, where z is a (global) conformal coordinate of A . Since M is complete with respect to ds^2 , we may assume that the restriction of ds^2 to A is complete on the set $\{z : |z| = r\}$, i.e., the set $\{z : |z| = r\}$ is at infinite distance from any point of A .

Let a^j ($1 \leq j \leq q$) be $q > 4$ distinct points in $\mathbb{P}^1(\mathbb{C})$. We may assume $a^j = (a_0^j : a_1^j)$ with $|a_0^j|^2 + |a_1^j|^2 = 1$ ($1 \leq j \leq q$), and we set $G_j := a_0^j g_1 - a_1^j g_0$ ($1 \leq j \leq q$) for the reduced representation $g = (g_0 : g_1)$ of the Gauss map. By the identity theorem, the G_j have at most countable many zeros. Let m_j be the limit inferior of the orders of the zeros of the functions G_j on A (and in particular $m_j = \infty$ if G_j has only a finite number of zeros on A).

All the m_j are increasing if we only consider the zeros which the functions G_j take on a subset $B \subset A$. So without loss of generality we may prove our theorem only on a sub-annular end, i.e. a subset $A_t := \{z : 0 < t \leq |z| < r < \infty\} \subset A$ with some t such that $1/r < t < r$. (We trivially observe that for $c := tr > 1$, $s := r/\sqrt{c}$, $\xi := z/\sqrt{c}$, we have $A_t = \{\xi : 0 < 1/s \leq |\xi| < s < \infty\}$.)

By passing to such a sub-annular end we will be able to extend the construction of a metric in step 2 below to the set $\{z : |z| = 1/r\}$, and, moreover, we may assume that for all $j = 1, \dots, q$:

$$\begin{aligned} &g \text{ omits } a^j \text{ (} m_j = \infty \text{) or takes } a^j \text{ infinitely often with ramification} \\ &2 \leq m_j < \infty \text{ and is ramified over } a^j \text{ with multiplicity at least } m_j. \end{aligned} \quad (3.3)$$

Step 2: On the annular end $A = \{z : 0 < 1/r \leq |z| < r < \infty\}$ minus a discrete subset $S \subset A$ we construct a flat metric $d\tau^2$ on $A \setminus S$ which is complete on the set $\{z : |z| = r\} \cup S$, i.e., the set $\{z : |z| = r\} \cup S$ is at infinite distance from any point of $A \setminus S$.

We may assume that

$$\sum_{j=1}^q \left(1 - \frac{1}{m_j}\right) > 4, \quad (3.4)$$

since otherwise Theorem 1 is already proved.

Take δ with

$$\frac{q - 4 - \sum_{j=1}^q \frac{1}{m_j}}{q} > \delta > \frac{q - 4 - \sum_{j=1}^q \frac{1}{m_j}}{q + 2},$$

and set $p = 2/(q - 2 - \sum_{j=1}^q \frac{1}{m_j} - q\delta)$. Then

$$0 < p < 1, \quad \frac{p}{1-p} > \frac{\delta p}{1-p} > 1. \quad (3.5)$$

Consider the subset

$$A_1 = A \setminus \{z : W_z(g_0, g_1)(z) = 0\}$$

of A . We define a new metric

$$d\tau^2 = |h|^{\frac{2p}{1-p}} \left(\frac{\prod_{j=1}^q |G_j|^{1 - \frac{1}{m_j} - \delta}}{|W(g_0, g_1)|} \right)^{\frac{2p}{1-p}} |dz|^2$$

on A_1 (where again $G_j := a_0^j g_1 - a_1^j g_0$ and h is defined with respect to the coordinate z on $A_1 \subset A$ and $W(g_0, g_1) = W_z(g_0, g_1)$) :

First we observe that $d\tau$ is continuous and nowhere vanishing on A_1 . Indeed, h is without zeros on A and for each $z_0 \in A_1$ with $G_j(z_0) \neq 0$ for all $j = 1, \dots, q$, $d\tau$ is continuous at z_0 .

Now, suppose there exists a point $z_0 \in A_1$ with $G_j(z_0) = 0$ for some j . Then $G_i(z_0) \neq 0$ for all $i \neq j$ and $\nu_{G_j}(z_0) \geq m_j \geq 2$. Changing the indices if necessary, we may assume that $g_0(z_0) \neq 0$, so also $a_0^j \neq 0$. So, we get

$$\nu_{W(g_0, g_1)}(z_0) = \nu_{\frac{(a_0^j g_1 - a_1^j)'(z_0)}{a_0^j}} = \nu_{\frac{(G_j/g_0)'(z_0)}{a_0^j}} = \nu_{G_j}(z_0) - 1 > 0. \quad (3.6)$$

This is in contradiction with $z_0 \in A_1$. Thus, $d\tau$ is continuous and nowhere vanishing on A_1 .

Next, it is easy to see that $d\tau$ is flat.

By Proposition 4 a) and the dependence of h on z and the independence of the G_j of z , we also easily see that $d\tau$ is independent of the choice of the coordinate z .

The key point is to prove the following claim :

Claim 1. $d\tau$ is complete on the set $\{z : |z| = r\} \cup \{z : W(g_0, g_1)(z) = 0\}$, i.e., the set $\{z : |z| = r\} \cup \{z : W(g_0, g_1)(z) = 0\}$ is at infinite distance from any interior point in A_1 .

If $W(g_0, g_1)(z_0) = 0$, then we have two cases :

Case 1. $G_j(z_0) = 0$ for some $j \in \{1, 2, \dots, q\}$.

Then we have $G_i(z_0) \neq 0$ for all $i \neq j$ and $\nu_{G_j}(z_0) \geq m_j$. By the same argument as in (3.6) we get that

$$\nu_{W(g_0, g_1)}(z_0) = \nu_{G_j}(z_0) - 1.$$

Thus (since $m_j \geq 2$),

$$\begin{aligned} \nu_{d\tau}(z_0) &= \frac{p}{1-p} \left(\left(1 - \frac{1}{m_j} - \delta\right) \nu_{G_j}(z_0) - \nu_{W(g_0, g_1)}(z_0) \right) \\ &= \frac{p}{1-p} \left(1 - \left(\frac{1}{m_j} + \delta\right) \nu_{G_j}(z_0) \right) \leq \frac{p}{1-p} \left(1 - \left(\frac{1}{m_j} + \delta\right) m_j \right) \\ &\leq -\frac{2\delta p}{1-p}. \end{aligned}$$

Case 2. $G_j(z_0) \neq 0$ for all $1 \leq j \leq q$.

It is easily to see that $\nu_{d\tau}(z_0) \leq -\frac{p}{1-p}$.

So, since $0 < \delta < 1$, we can find a positive constant C such that

$$|d\tau| \geq \frac{C}{|z - z_0|^{\delta p/(1-p)}} |dz|$$

in a neighborhood of z_0 . Combining with (3.5) we thus have that $d\tau$ is complete on $\{z : W(g_0, g_1)(z) = 0\}$.

Now assume that $d\tau$ is not complete on $\{z : |z| = r\}$. Then there exists $\gamma : [0, 1) \rightarrow A_1$, where $\gamma(1) \in \{z : |z| = r\}$, so that $|\gamma| < \infty$. Furthermore, we may also assume that $\text{dist}(\gamma(0); \{z : |z| = 1/r\}) > 2|\gamma|$. Consider a small disk Δ with center at $\gamma(0)$. Since $d\tau$ is flat, Δ is isometric to an ordinary disk in the plane (cf. e.g. Lemma 9). Let $\Phi : \{|w| < \eta\} \rightarrow \Delta$ be this isometry. Extend Φ , as a local isometry into A_1 , to the largest disk $\{|w| < R\} = \Delta_R$ possible. Then $R \leq |\gamma|$. The reason that Φ cannot be extended to a larger disk is that the image goes to the outside boundary $\{z : |z| = r\}$ of A_1 (it cannot go to points of A with $W(g_0, g_1) = 0$ since we have shown already the completeness of A_1 with respect to these points). More precisely, there exists a point w_0 with $|w_0| = R$ so that $\Phi(\overline{0, w_0}) = \Gamma_0$ is a divergent curve on A . The map $\Phi(w)$ is locally biholomorphic, and the metric on Δ_R induced from ds^2 through Φ is given by

$$\Phi^* ds^2 = |h \circ \Phi|^2 |g \circ \Phi|^4 \left| \frac{dz}{dw} \right|^2 |dw|^2. \quad (3.7)$$

On the other hand, Φ is isometric, so we have

$$|dw| = |d\tau| = \left(\frac{|h| \prod_{j=1}^q |G_j|^{(1 - \frac{1}{m_j} - \delta)p}}{|W(g_0, g_1)|^p} \right)^{\frac{1}{1-p}} |dz|$$

$$\Rightarrow \left| \frac{dw}{dz} \right|^{1-p} = \frac{|h| \prod_{j=1}^q |G_j|^{(1-\frac{1}{m_j}-\delta)p}}{|W(g_0, g_1)|^p}.$$

Set $f := g(\Phi)$, $f_0 := g_0(\Phi)$, $f_1 := g_1(\Phi)$ and $F_j := G_j(\Phi)$. Since

$$W_w(f_0, f_1) = (W_z(g_0, g_1) \circ \Phi) \frac{dz}{dw},$$

we obtain

$$\left| \frac{dz}{dw} \right| = \frac{|W(f_0, f_1)|^p}{|h(\Phi)| \prod_{j=1}^q |F_j|^{(1-\frac{1}{m_j}-\delta)p}} \quad (3.8)$$

By (3.7) and (3.8) and by definition of p , therefore, we get

$$\begin{aligned} \Phi^* ds^2 &= \left(\frac{\|f\|^2 |W(f_0, f_1)|^p}{\prod_{j=1}^q |F_j|^{(1-\frac{1}{m_j}-\delta)p}} \right)^2 |dw|^2 \\ &= \left(\frac{\|f\|^{q-2-\sum_{j=1}^q \frac{1}{m_j}-q\delta} |W(f_0, f_1)|}{\prod_{j=1}^q |F_j|^{1-\frac{1}{m_j}-\delta}} \right)^{2p} |dw|^2. \end{aligned}$$

Using the Lemma 8, we obtain

$$\Phi^* ds^2 \leq C_0^{2p} \cdot \left(\frac{2R}{R^2 - |w|^2} \right)^{2p} |dw|^2.$$

Since $0 < p < 1$, it then follows that

$$d_{\Gamma_0} \leq \int_{\Gamma_0} ds = \int_{\overline{0, w_0}} \Phi^* ds \leq C_0^p \cdot \int_0^R \left(\frac{2R}{R^2 - |w|^2} \right)^p |dw| < +\infty,$$

where d_{Γ_0} denotes the length of the divergent curve Γ_0 in M , contradicting the assumption of completeness of M . Claim 1 is proved.

To summarize, in step 2 we have constructed, for $A = \{z : 0 < 1/r \leq |z| < r < \infty\}$ and $S = \{z : W_z(g_0, g_1)(z) = 0\}$, a continuous and nowhere vanishing metric $d\tau^2$ on $A \setminus S$ which is flat, independent of the choice of coordinate z , and complete with respect to the points of S and with respect to the (outside) boundary $\{z : |z| = r\}$.

Step 3: We will "symmetrize" the metric constructed in step 2 so that it will become a complete and flat metric on $Int(A) \setminus (S \cup \tilde{S})$ (with \tilde{S} another discrete subset).

We introduce a new coordinate $\xi(z) := 1/z$. By Proposition 4 a) we have $S = \{z : W_z(g_0, g_1)(z) = 0\} = \{z : W_\xi(g_0, g_1)(z) = 0\}$ (where the

zeros are taken with the same multiplicities) and since $d\tau^2$ is independent of the coordinate z , the change of coordinate $\xi(z) = 1/z$ yields an isometry of $A \setminus S$ onto the set $\tilde{A} \setminus \tilde{S}$, where $\tilde{A} := \{z : 1/r < |z| \leq r\}$ and $\tilde{S} := \{z : W_z(g_0, g_1)(1/z) = 0\}$. In particular we have (if still \tilde{h} is defined with respect to the coordinate ξ) :

$$\begin{aligned} d\tau^2 &= |\tilde{h}(1/z)|^{\frac{2}{1-p}} \left(\frac{\prod_{j=1}^q |G_j(1/z)|^{1-\frac{1}{m_j}-\delta}}{|W_{(1/z)}(g_0, g_1)(1/z)|} \right)^{\frac{2p}{1-p}} |d(1/z)|^2 \\ &= \left(|h(1/z)|^{\frac{2}{1-p}} \left(\frac{\prod_{j=1}^q |G_j(1/z)|^{1-\frac{1}{m_j}-\delta}}{|W_z(g_0, g_1)(1/z)|} \right)^{\frac{2p}{1-p}} \left| \frac{dz}{d(1/z)} \right|^2 \right) |d(1/z)|^2 \\ &= |h(1/z)|^{\frac{2}{1-p}} \left(\frac{\prod_{j=1}^q |G_j(1/z)|^{1-\frac{1}{m_j}-\delta}}{|W_z(g_0, g_1)(1/z)|} \right)^{\frac{2p}{1-p}} |dz|^2 \end{aligned}$$

We now define

$$\begin{aligned} d\tilde{\tau}^2 &= \left(|h(z)h(1/z)| \cdot \frac{\prod_{j=1}^q |G_j(z)G_j(1/z)|^{(1-\frac{1}{m_j}-\delta)p}}{|W_z(g_0, g_1)(z) \cdot W_z(g_0, g_1)(1/z)|^p} \right)^{\frac{2}{1-p}} |dz|^2 \\ &= \lambda^2(z) |dz|^2, \end{aligned}$$

on $\tilde{A}_1 := \{z : 1/r < |z| < r\} \setminus \{z : W_z(g_0, g_1)(z) \cdot W_z(g_0, g_1)(1/z) = 0\}$. Then $d\tilde{\tau}^2$ is complete on \tilde{A}_1 : In fact by what we showed above we have: Towards any point of the boundary $\partial\tilde{A}_1 := \{z : 1/r = |z|\} \cup \{z : |z| = r\} \cup \{z : W_z(g_0, g_1)(z) \cdot W_z(g_0, g_1)(1/z) = 0\}$ of \tilde{A}_1 , one of the factors of $\lambda^2(z)$ is bounded from below away from zero, and the other factor is the one of a complete metric with respect of this part of the boundary. Moreover by the corresponding properties of the two factors of $\lambda^2(z)$ it is trivial that $d\tilde{\tau}^2$ is a continuous nowhere vanishing and flat metric on \tilde{A}_1 .

Step 4 : We produce a contradiction by using Lemma 9 to the open Riemann surface $(\tilde{A}_1, d\tilde{\tau}^2)$:

In fact, we apply Lemma 9 to any point $p \in \tilde{A}_1$. Since $d\tilde{\tau}^2$ is complete, there cannot exist a divergent curve from p to the boundary $\partial\tilde{A}_1$ with finite length with respect to $d\tilde{\tau}^2$. Since $\Phi : \Delta_{R_0} \rightarrow \tilde{A}_1$ is a local isometry, we necessarily have $R_0 = \infty$. So $\Phi : \mathbb{C} \rightarrow \tilde{A}_1 \subset \{z : |z| < r\}$ is a non constant holomorphic map, which contradicts to Liouville's theorem. So our assumption (3.4) was wrong. This proves the Theorem 1. \square

4. THE PROOF OF THEOREM 3

Proof. For convenience of the reader, we first recall some notations on the Gauss map of minimal surfaces in \mathbb{R}^4 . Let $x = (x_1, x_2, x_3, x_4) : M \rightarrow \mathbb{R}^4$ be a non-flat complete minimal surface in \mathbb{R}^4 . As is well-known, the set of all oriented 2-planes in \mathbb{R}^4 is canonically identified with the quadric

$$Q_2(\mathbb{C}) := \{(w_1 : \dots : w_4) | w_1^2 + \dots + w_4^2 = 0\}$$

in $\mathbb{P}^3(\mathbb{C})$. By definition, the Gauss map $g : M \rightarrow Q_2(\mathbb{C})$ is the map which maps each point p of M to the point of $Q_2(\mathbb{C})$ corresponding to the oriented tangent plane of M at p . The quadric $Q_2(\mathbb{C})$ is biholomorphic to $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. By suitable identifications we may regard g as a pair of meromorphic functions $g = (g^1, g^2)$ on M . Let z be a local holomorphic coordinate. Set $\phi_i := \partial x_i / dz$ for $i = 1, \dots, 4$. Then, g^1 and g^2 are given by

$$g^1 = \frac{\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2}, \quad g^2 = \frac{-\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2}$$

and the metric on M induced from \mathbb{R}^4 is given by

$$ds^2 = |\phi|^2(1 + |g^1|^2)(1 + |g^2|^2)|dz|^2,$$

where $\phi := \phi_1 - \sqrt{-1}\phi_2$. We remark that although the ϕ_i , ($i = 1, 2, 3, 4$) and ϕ depend on z , $g = (g^1, g^2)$ and ds^2 do not. Next we take reduced representations $g^l = (g_0^l : g_1^l)$ on M and set $\|g^l\| = (|g_0^l|^2 + |g_1^l|^2)^{1/2}$ for $l = 1, 2$. Then we can rewrite

$$ds^2 = |h|^2 \|g^1\|^2 \|g^2\|^2 |dz|^2, \quad (4.9)$$

where $h := \phi / (g_0^1 g_0^2)$. In particular, h is a holomorphic map without zeros. We remark that h depends on z , however, the reduced representations $g^l = (g_0^l : g_1^l)$ are globally defined on M and independent of z . Finally we observe that by the assumption that M is not flat, g is not constant.

Now the proof of Theorem 3 will be given in four steps :

Step 1: This step is completely analogue to step 1 in the proof of Theorem 1. We get : By passing to a sub-annular end we may assume

that the annular end is $A = \{z : 0 < 1/r \leq |z| < r < \infty\}$, where z is a (global) conformal coordinate of A , that the restriction of ds^2 to A is complete on the set $\{z : |z| = r\}$, i.e., the set $\{z : |z| = r\}$ is at infinite distance from any point of A , and, moreover, that for all $j = 1, \dots, q_l$, $l = 1, 2$ (case (i)) respectively for all $j = 1, \dots, q_1$, $l = 1$ (case (ii)), we have :

$$g^l \text{ omits } a^{lj} \text{ (} m_{lj} = \infty \text{) or takes } a^{lj} \text{ infinitely often with ramification } 2 \leq m_{lj} < \infty \text{ and is ramified over } a^{lj} \text{ with multiplicity at least } m_{lj}. \quad (4.10)$$

From now on we separate the two cases (i) and (ii), dealing first with the case (i).

Step 2 for the case (i): Our strategy is the same as for step 2 in the proof of Theorem 1. We may assume that $\gamma_1 = \sum_{j=1}^{q_1} (1 - \frac{1}{m_{1j}}) > 2$, $\gamma_2 = \sum_{j=1}^{q_2} (1 - \frac{1}{m_{2j}}) > 2$, and

$$\frac{1}{\gamma_1 - 2} + \frac{1}{\gamma_2 - 2} < 1, \quad (4.11)$$

since otherwise case (i) of Theorem 3 is already proved.

Choose $\delta_0 (> 0)$ such that $\gamma_l - 2 - q_l \delta_0 > 0$ for all $l = 1, 2$, and

$$\frac{1}{\gamma_1 - 2 - q_1 \delta_0} + \frac{1}{\gamma_2 - 2 - q_2 \delta_0} = 1.$$

If we choose a positive constant $\delta (< \delta_0)$ sufficiently near to δ_0 and set

$$p_l := 1/(\gamma_l - 2 - q_l \delta), \quad (l = 1, 2),$$

we have

$$0 < p_1 + p_2 < 1, \quad \frac{\delta p_l}{1 - p_1 - p_2} > 1 \quad (l = 1, 2). \quad (4.12)$$

Consider the subset

$$A_2 = A \setminus \{z : W_z(g_0^1, g_1^1)(z) \cdot W_z(g_0^2, g_1^2)(z) = 0\}$$

of A . We define a new metric

$$d\tau^2 = \left(|h| \frac{\prod_{j=1}^{q_1} |G_j^1|^{(1 - \frac{1}{m_{1j}} - \delta)p_1} \prod_{j=1}^{q_2} |G_j^2|^{(1 - \frac{1}{m_{2j}} - \delta)p_2}}{|W(g_0^1, g_1^1)|^{p_1} |W(g_0^2, g_1^2)|^{p_2}} \right)^{\frac{2}{1 - p_1 - p_2}} |dz|^2$$

on A_2 (where again $G_j^l := a_0^{lj} g_1^l - a_1^{lj} g_0^l$ ($l = 1, 2$) and h is defined with respect to the coordinate z on $A_2 \subset A$ and $W(g_0^l, g_1^l) = W_z(g_0^l, g_1^l)$).

It is easy to see that by the same arguments as in step 2 of the proof of Theorem 1 (applied for each $l = 1, 2$), we get that $d\tau$ is a continuous nowhere vanishing and flat metric on A_2 , which is moreover independant of the choice of the coordinate z .

The key point is to prove the following claim :

Claim 2. $d\tau^2$ is complete on the set $\{z : |z| = r\} \cup \{z : \Pi_{l=1,2}W(g_0^l, g_1^l)(z) = 0\}$, i.e., the set $\{z : |z| = r\} \cup \{z : \Pi_{l=1,2}W(g_0^l, g_1^l)(z) = 0\}$ is at infinite distance from any interior point in A_2 .

It is easy to see that by the same method as in the proof of Claim 1 in the proof of Theorem 1, we may show that $d\tau$ is complete on $\{z : \Pi_{l=1,2}W(g_0^l, g_1^l)(z) = 0\}$.

Now assume $d\tau$ is not complete on $\{z : |z| = r\}$. Then there exists $\gamma : [0, 1) \rightarrow A_2$, where $\gamma(1) \in \{z : |z| = r\}$, so that $|\gamma| < \infty$. Furthermore, we may also assume that $\text{dist}(\gamma(0), \{z : |z| = 1/r\}) > 2|\gamma|$. Consider a small disk Δ with center at $\gamma(0)$. Since $d\tau$ is flat, Δ is isometric to an ordinary disk in the plane. Let $\Phi : \{|w| < \eta\} \rightarrow \Delta$ be this isometry. Extend Φ , as a local isometry into A_2 , to the largest disk $\{|w| < R\} = \Delta_R$ possible. Then $R \leq |\gamma|$. The reason that Φ cannot be extended to a larger disk is that the image goes to the outside boundary $\{z : |z| = r\}$ of A_2 . More precisely, there exists a point w_0 with $|w_0| = R$ so that $\Phi(\overline{0, w_0}) = \Gamma_0$ is a divergent curve on A . The map $\Phi(w)$ is locally biholomorphic, and the metric on Δ_R induced from ds^2 through Φ is given by

$$\Phi^* ds^2 = |h \circ \Phi|^2 |g^1 \circ \Phi|^2 |g^2 \circ \Phi|^2 \left| \frac{dz}{dw} \right|^2 |dw|^2. \quad (4.13)$$

On the other hand, Φ is isometric, so we have

$$\begin{aligned} |dw| &= |d\tau| = \left(|h| \frac{\prod_{j=1}^{q_1} |G_j^1|^{(1-\frac{1}{m_{1j}}-\delta)p_1} \prod_{j=1}^{q_2} |G_j^2|^{(1-\frac{1}{m_{2j}}-\delta)p_2}}{|W(g_0^1, g_1^1)|^{p_1} |W(g_0^2, g_1^2)|^{p_2}} \right)^{\frac{1}{1-p_1-p_2}} |dz| \\ \Rightarrow \left| \frac{dw}{dz} \right|^{1-p_1-p_2} &= |h| \frac{\prod_{j=1}^{q_1} |G_j^1|^{(1-\frac{1}{m_{1j}}-\delta)p_1} \prod_{j=1}^{q_2} |G_j^2|^{(1-\frac{1}{m_{2j}}-\delta)p_2}}{|W(g_0^1, g_1^1)|^{p_1} |W(g_0^2, g_1^2)|^{p_2}}. \end{aligned}$$

For each $l = 1, 2$, we set $f^l := g^l(\Phi)$, $f_0^l := g_0^l(\Phi)$, $f_1^l := g_1^l(\Phi)$ and $F_j^l := G_j^l(\Phi)$. Since

$$W_w(f_0^l, f_1^l) = (W_z(g_0^l, g_1^l) \circ \Phi) \frac{dz}{dw}, \quad (l = 1, 2),$$

we obtain

$$\left| \frac{dz}{dw} \right| = \frac{\prod_{l=1,2} |W(f_0^l, f_1^l)|^{p_l}}{|h(\Phi)| \prod_{l=1,2} \prod_{j=1}^{q_l} |F_j^l|^{(1-\frac{1}{m_{lj}}-\delta)p_l}}. \quad (4.14)$$

By (4.13) and (4.14), we get

$$\begin{aligned} \Phi^* ds^2 &= \left(\prod_{l=1,2} \frac{\|f^l\| (|W(f_0^l, f_1^l)|)^{p_l}}{\prod_{j=1}^{q_l} |F_j^l|^{(1-\frac{1}{m_{lj}}-\delta)p_l}} \right)^2 |dw|^2 \\ &= \prod_{l=1,2} \left(\frac{\|f^l\|^{q_l-2-\sum_{j=1}^{q_l} \frac{1}{m_{lj}}-q_l\delta} |W(f_0^l, f_1^l)|}{\prod_{j=1}^{q_l} |F_j^l|^{1-\frac{1}{m_{lj}}-\delta}} \right)^{2p_l} |dw|^2. \end{aligned}$$

Using the Lemma 8, we obtain

$$\Phi^* ds^2 \leq C_0^{2(p_1+p_2)} \cdot \left(\frac{2R}{R^2 - |w|^2} \right)^{2(p_1+p_2)} |dw|^2.$$

Since $0 < p_1 + p_2 < 1$ by (4.12), it then follows that

$$d_{\Gamma_0} \leq \int_{\Gamma_0} ds = \int_{0, w_0} \Phi^* ds \leq C_0^{p_1+p_2} \cdot \int_0^R \left(\frac{2R}{R^2 - |w|^2} \right)^{p_1+p_2} |dw| < +\infty,$$

where d_{Γ_0} denotes the length of the divergent curve Γ_0 in M , contradicting the assumption of completeness of M . Claim 2 is proved.

Steps 3 and 4 for the case (i): These steps are analogue to the corresponding steps in the proof of Theorem 1. Define $d\tilde{\tau}^2 = \lambda^2(z)|dz|^2$ on

$$\tilde{A}_2 := \{z : 1/r < |z| < r\} \setminus$$

$$\setminus \{z : W_z(g_0^1, g_1^1)(z) \cdot W_z(g_0^2, g_1^2)(z) \cdot W_z(g_0^1, g_1^1)(1/z) \cdot W_z(g_0^2, g_1^2)(1/z) = 0\},$$

where

$$\begin{aligned} \lambda(z) &= \left(|h(z)| \frac{\prod_{j=1}^{q_1} |G_j^1(z)|^{(1-\frac{1}{m_{1j}}-\delta)p_1} \prod_{j=1}^{q_2} |G_j^2(z)|^{(1-\frac{1}{m_{2j}}-\delta)p_2}}{|W_z(g_0^1, g_1^1)(z)|^{p_1} |W_z(g_0^2, g_1^2)(z)|^{p_2}} \right)^{\frac{1}{1-p_1-p_2}} \\ &\times \left(|h(1/z)| \frac{\prod_{j=1}^{q_1} |G_j^1(1/z)|^{(1-\frac{1}{m_{1j}}-\delta)p_1} \prod_{j=1}^{q_2} |G_j^2(1/z)|^{(1-\frac{1}{m_{2j}}-\delta)p_2}}{|W_z(g_0^1, g_1^1)(1/z)|^{p_1} |W_z(g_0^2, g_1^2)(1/z)|^{p_2}} \right)^{\frac{1}{1-p_1-p_2}}. \end{aligned}$$

By using Claim 2, the continuous nowhere vanishing and flat metric $d\tau$ on A_2 is also complete. Using the identical argument of step 4 in the proof of Theorem 1 to the open Riemann surface $(\tilde{A}_2, d\tilde{\tau})$ produces a contradiction, so assumption (4.11) was wrong. This implies case (i) of the Theorem 3.

We finally consider the case (ii) of Theorem 3 (where $g^2 \equiv \text{constant}$ and $g^1 \not\equiv \text{constant}$). Suppose that $\gamma_1 > 3$. We can choose δ with

$$\frac{\gamma_1 - 3}{q_1} > \delta > \frac{\gamma_1 - 3}{q_1 + 1},$$

and set $p = 1/(\gamma_1 - 2 - q_1\delta)$. Then

$$0 < p < 1, \quad \frac{p}{1-p} > \frac{\delta p}{1-p} > 1.$$

Set

$$d\tau^2 = |h|^{\frac{2}{1-p}} \left(\frac{\prod_{j=1}^{q_1} |G_j^1|^{1-\frac{1}{m_{1j}}-\delta}}{|W(g_0^1, g_1^1)|} \right)^{\frac{2p}{1-p}} |dz|^2.$$

Using this metric, by the analogue arguments as in step 2 to step 4 of the proof of Theorem 1, we get the case (ii) of Theorem 3. \square

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