Logarithmic Jet Bundles

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0 Introduction

Jet bundles have turned out to be a powerful tool in geometry. Green and Griffiths [6] explained, for example, an approach to establish Bloch's theorem on the algebraic degeneracy of holomorphic maps into abelian varieties by constructing negatively curved pseudometrics on jet bundles and by applying Ahlfors' lemma. This approach was clarified by Siu and Yeung ([11]), after the same authors gave a proof of this result using jet bundles and value distribution theory ([9]). Demailly ([1]) presented a new construction of projective jets and pseudo-metrics on them. These projective jets are closer to the geometry of holomorphic curves than the usual jets, since the action of the group of reparametrizations of germs of curves, which is geometrically redundant, is divided out. Using these pseudometrics on projective jets Demailly and El Goul ([2], [3]) were able to show that a generic surface X in $\mathbf{P}^3$ of degree $d \geq 42$ is Kobayashi hyperbolic. As a corollary one obtains that a generic curve in $\mathbf{P}^2$ of degree $d \geq 42$ is hyperbolic and hyperbolically embedded, a result which was previously proved by Siu and Yeung ([10]) for much higher degree, using jet bundles and value distribution theory.
It was conjectured by Kobayashi that a generic curve in $\mathbf{P}^2$ of degree $d \geq 5$ is hyperbolic and hyperbolically embedded. One of the reasons that the degrees reached by Siu and Yeung and by Demailly and El Goul are only much higher is probably due to their indirect approach, namely that they prove the hyperbolicity of a surface in $X \subset \mathbf{P}^3$ such that the curve in $\mathbf{P}^2$ is just the branching divisor of a projection of X to $\mathbf{P}^2$.
This shows that it desirable to establish a technique which works directly on the quasi-projective case. For jet bundles this was done already by Noguchi ([8]). The purpose of the present talk, which gives an outline of the first part of our work [4], is to show how one can generalize Demailly's approach of projective jets to the logarithmic case. We also have another, more intrinsic way to obtain this generalization, see [5].

1 Log-directed jet bundles

1.1 Logarithmic jet bundles

In this subsection we mainly follow Noguchi ([8]).

Let X be a complex manifold. Let $x \in X$. We consider germs $f : (\mathbf{C}, 0) \to (X, x)$ of holomorphic curves through x. Two such germs are considered to be equivalent if they have the same Taylor expansions of order k in some (and hence any) local coordinate around x. Denote the equivalence class by $j_k(f)$. Then we define $J_kX_x = \{j_k(f) | f : (\mathbf{C}, 0) \to (X, x)\}$ and $J_kX = \bigcup_{x \in X} J_kX_x$. Let $\pi : J_kX \to X$ be the natural projection. Then J_kX carries the structure of a holomorphic fiber bundle over X. It is called the *k-jet bundle* over X. There exist, for $k \geq l$, canonical projection maps

$$\pi_{k,l} : J_kX \to J_lX \; ; \;\; j_k(f) \to j_l(f), \tag{1.1}$$

and J_1X is canonically isomorphic to TX. If $F : X \to Y$ is a holomorphic map to another complex manifold Y, then it induces a holomorphic map

$$F_k : J_kX \to J_kY \; ; \;\; j_k(f) \to j_k(F \circ f) \tag{1.2}$$

over F.

Let ΩX be the sheaf of germs of holomorphic 1-forms on X. Take a holomorphic section $\omega \in H^0(\Omega X, U)$ for some open subset $U \subset X$. For $j_k(f) \in J_kX|_U$ we put $f^*\omega = Z(t)dt$. Then the derivatives $\frac{d^jZ}{dt^j}$, $0 \leq j \leq k-1$ are well defined, independantly of the representative f for $j_k(f)$. Hence, we have a well defined mapping

$$\tilde{\omega} : J_kX|_U \to \mathbf{C}^k \; ; \;\; j_k(f) \to (\frac{d^jZ}{dt^j}(0))_{0 \leq j \leq k-1} \tag{1.3}$$

which is holomorphic. If, moreover, ω^1,...,ω^n with $n = \dim X$ are holomorphic 1-forms on U such that $\omega^1 \wedge ... \wedge \omega^n$ does not vanish anywhere, then we have a biholomorphic map

$$\pi \times (\tilde{\omega}^1, ..., \tilde{\omega}^n) : J_kX|_U \to U \times (\mathbf{C}^k)^n \tag{1.4}$$

which we call the *trivialization* associated with ω^1,...,ω^n. If ω is only a section over U in the sheaf of meromorphic 1-forms, then the map $\tilde{\omega}$ defined as in equation (1.3) induces still a meromorphic vector function

$$\tilde{\omega} : J_kX|_U \to \mathbf{C}^k \tag{1.5}$$

Let $\bar{X}$ be a complex manifold with a normal crossing divisor D. This means that around any point x of $\bar{X}$, there exist local coordinates $z_1, ..., z_n$ centered at x such that D is defined by $z_1 z_2 ... z_l = 0$ in this coordinate neighborhood for some $l \leq n$. The pair $(\bar{X}, D)$ will be called a *log-manifold.* Let $X = \bar{X} \setminus D$.

We define the logarithmic cotangent sheaf $\bar{\Omega}X$ as the locally free subsheaf of the sheaf of meromorphic 1-forms on $\bar{X}$, whose restriction to X is identical to ΩX and whose localization at any point $x \in D$, in local coordinates $z_1, ..., z_n$ around x as above, is given by

$$\bar{\Omega}X_x = \sum_{i=1}^{l} \mathcal{O}_{\bar{X},x} \frac{dz_i}{z_i} + \sum_{j=l+1}^{n} \mathcal{O}_{\bar{X},x} dz_j, \tag{1.6}$$

Its dual is called the logarithmic tangent sheaf $\bar{T}X$. It is a locally free subsheaf of the holomorphic tangent bundle TX over X.

Given log-manifolds $(\bar{X}', D')$ and $(\bar{X}, D)$, a holomorphic map $F : \bar{X}' \to \bar{X}$ such that $F^{-1}D \subset D'$ will be called a *log-morphism* $F : X' \to X$ A log-morphism $F : X' \to X$ induces vector bundle morphisms (see Iitaka [7]),

$$F^* : \bar{\Omega}X \to \bar{\Omega}X', F_* : \bar{T}X' \to \bar{T}X. \tag{1.7}$$

Let $s \in H^0(J_k\bar{X}, U)$ be a holomorphic section over an open subset $U \subset \bar{X}$. We say that s is a *logarithmic jet field* if the maps $\tilde{\omega} \circ s|_{U'} : U' \to \mathbf{C}^k$ is holomorphic for all $\omega \in H^0(\bar{\Omega}X, U')$, where U' is an open subset of U and the map $\tilde{\omega}$ is defined like in equation (1.5).

The sets of logarithmic jet fields over open subsets of X define a subsheaf of the sheaf $J_k\bar{X}$, which we denote by $\bar{J}_kX$ and which we call *logarithmic k-jet bundle* over X.

Proposition 1.1 .
a) The sheaf $\bar{J}_kX$ has the structure of a locally trivial fiber bundle over $\bar{X}$.
b) We have a canonical identification of $(\bar{J}_kX)|_X$ with J_kX.
c) There exist, for $k \geq l$, canonical projection maps $\pi_{k,l} : \bar{J}_kX \to \bar{J}_lX$, which extends the map $(\pi_{k,l}|_{J_kX}) : J_kX \to J_lX$ (c.f. equation (1.1)), and $\bar{J}_1X$ is canonically isomorphic to $\bar{T}X$.
d) A log-morphism $F : X' \to X$ induces a canonical map $F_k : \bar{J}_kX' \to \bar{J}_kX$, which extends the map $F_k|_{J_kX} : J_kX' \to J_kX$ (c.f. equation (1.2)).

One can express the local triviality of $\bar{J}_kX$ explicitly in terms of coordinates $z_1, ..., z_n$ in $U \subset \bar{X}$ in which $D = \{z_1z_2...z_l = 0\}$. Let $\omega^1 = \frac{dz_1}{z_1}, ..., \omega^l = \frac{dz_l}{z_l}$, $\omega^{l+1} = dz_{l+1}, ..., \omega^n = dz_n$. Then we have a biholomorphic map (c.f. also equations (1.4) and (1.5) and the definition of $\bar{J}_kX$)

$$\pi \times (\tilde{\omega}^1, ..., \tilde{\omega}^n) : \bar{J}_kX|_U \to U \times (\mathbf{C}^k)^n. \tag{1.8}$$

1.2 Log-directed jet bundles

We first follow Demailly ([1]). Let X be a complex manifold together with a holomorphic subbundle $V \subset TX$. The pair (X, V) is called a *complex directed manifold.* If (X, V) and (Y, W) are two complex directed manifolds, then a holomorphic map $F : X \to Y$ which satisfies $F_*(V) \subset W$ is called *directed morphism.*

Let (X, V) be a directed manifold. The *directed k-jet bundle* over (X, V) is defined to be the subset of k-jets $j_k(f) \in J_kX$ of germs $f : (\mathbf{C}, 0) \to (X, x)$ such that $f'(t) \in V_{f(t)}$ for all t in a neighborhood of 0. If $F : (X, V) \to (Y, W)$ is a directed morphism, then equation (1.2) induces a holomorphic map

$$F_k : J_kV \to J_kW \ ; \ \ j_k(f) \to j_k(F \circ f) \tag{1.9}$$

over F, since the restriction of $F_k : J_kX \to J_kY$ to J_kV maps to J_kW.

We now generalize Demailly's concept of directed k-jet bundles to the logarithmic context. We define a *log-directed manifold* to be the triple $(\bar{X}, D, \bar{V})$, where $(\bar{X}, D)$ is a log-manifold together with a subbundle $\bar{V}$ of $\bar{T}X$. A *log-directed morphism* between log-directed manifolds $(\bar{X}', D', \bar{V}')$ and $(\bar{X}, D, \bar{V})$ is a log-morphism $F : (\bar{X}', D') \to (\bar{X}, D)$ such that $F_*(\bar{V}') \subset \bar{V}$, where F_* is defined as in equation (1.7).

Let $(\bar{X}, D, \bar{V})$ be a log-directed manifold. In order to simplify notation in all what follows we put $V = \bar{V}|_X$. By Proposition 1.1 we can canonically identify $(\bar{J}_kX)|_X$ with J_kX, and by definition we have $(\bar{T}X)|X = TX$. Hence, the directed k-jet bundle J_kV over (X, V) can be considered as a subset of logarithmic k-jet bundle $\bar{J}_kX$ over $\bar{X}$. We define the *log-directed k-jet bundle* $\bar{J}_kV$ over $(\bar{X}, D, \bar{V})$ to be the closure $\overline{J_kV} \subset \bar{J}_kX$ of J_kV in $\bar{J}_kX$. If $F : (\bar{X}', D', \bar{V}') \to (\bar{X}, D, \bar{V})$ is a log-directed morphism, it induces a holomorphic map

$$F_k : \bar{J}_kV' \to \bar{J}_kV \tag{1.10}$$

over F which is the restriction of the canonical map $F_k : \bar{J}_kX \to \bar{J}_kX'$ (c.f. Proposition 1.1) and extends the map $F_k|_X : J_kV' \to J_kV$ (c.f. equation (1.9)).

Proposition 1.2 *Let $(\bar{X}, D, \bar{V})$ be a log-directed manifold.*
a) For any $x \in X$ there exists a neighborhood U of x in $\bar{X}$ and a log-directed projection

$$K : (\bar{X}, D, \bar{V})|_U \to (\mathbf{C}^r, E, \bar{T}\mathbf{C}^r) \ ; \ \ (z_1, ..., z_n) \to (z_1, ..., z_r),$$

with $E = \{z_1...z_a = 0\}$ and $a \leq r = \mathrm{rank} V$, which has bijective differential K_* for all points $y \in U$. Over U it induces a bundle isomorphism

$$K_k : \bar{J}_kV \to K^{-1}(\bar{J}_k(\mathbf{C}^r \setminus E)).$$

b) $\bar{J}_kV \subset \bar{J}_kX$ is a submanifold of $\bar{J}_kX$. □

Let (X, V) be a directed manifold. The subset $J_kV^{\text{sing}} \subset J_kV$ of *singular k-jets* is defined to be the subset of k-jets $j_k(f) \in J_kV$ of germs $f : (\mathbf{C}, 0) \to (X, x)$ such that $f'(0) = 0$. The subset $J_kV^{\text{reg}} \subset J_kV$ of *regular k-jets* is defined to be the complement $J_kV^{\text{reg}} = J_kV \setminus J_kV^{\text{sing}}$.

Let now $(\bar{X}, D, \bar{V})$ be a log-directed manifold. We define $\bar{J}_kV^{\text{sing}} \subset \bar{J}_kV$ to be the closure $\overline{J_kV^{\text{sing}}} \subset \bar{J}_kV$ of J_kV^{sing} in $\bar{J}_kV$, and we define $\bar{J}_kV^{\text{reg}} = \bar{J}_kV \setminus \bar{J}_kV^{\text{sing}}$.

Proposition 1.3 *$\bar{J}_kV^{\text{sing}} \subset \bar{J}_kV$ is a smooth submanifold of codimension $r = \text{rank}\bar{V}$.* □

2 Logarithmic Demailly-Semple jet bundles

We begin with a log-directed manifold $X_0 = (\bar{X}_0, D_0, \bar{V}_0)$. We define $(\bar{X}_k, D_k, \bar{V}_k)$ inductively as follows. Let $\bar{X}_k = \mathbf{P}(\bar{V}_{k-1})$ with its natural projection π_k to $\bar{X}_{k-1}$. Set $D_k = \pi_k^{-1}D_{k-1}$ and $X_k = \bar{X}_k \backslash D_k$. Let $\mathcal{O}_{\bar{X}_k}(-1)$ be the tautological subbundle of $\pi_k^{-1}\bar{V}_{k-1} \subseteq \pi_k^{-1}\bar{T}X_{k-1}$, and set

$$\bar{V}_k = (\pi_k)_*^{-1}\Big(\mathcal{O}_{\bar{X}_k}(-1)\Big). \tag{2.1}$$

Since $(\pi_k)_* : \bar{T}X_k \to \pi_k^{-1}\bar{T}X_{k-1}$ has maximal rank everywhere as it is a bundle projection, we see that $\bar{V}_k$ is a subbundle of $\bar{T}X_k$ giving a log-directed structure for X_k and also for π_k, thus completing our inductive definition.

We set $P_kV = X_k$, $\bar{P}_kV = \bar{X}_k$ and $P_kX = P_kTX$, $\bar{P}_kX = \bar{P}_kTX$. We also set (for $j < k$) $\pi_{j,k} = \pi_{j+1} \circ \cdots \circ \pi_{k-1} \circ \pi_k : \bar{P}_kV \to \bar{P}_jV$.

Note that $\ker(\pi_k)_* = T_{\bar{P}_kV/\bar{P}_{k-1}V}$ by definition. Hence, we have the following short exact sequence of vector bundles over $\bar{P}_kV$:

$$0 \longrightarrow T_{\bar{P}_kV/\bar{P}_{k-1}V} \longrightarrow \bar{V}_k \xrightarrow{(\pi_k)_*} \mathcal{O}_{\bar{P}_kV}(-1) \longrightarrow 0. \tag{2.2}$$

We now assume $r \geq 2$ throughout, for the situation is trivial otherwise. The composition of vector bundle morphisms over $\bar{P}_kV$

$$\mathcal{O}_{\bar{P}_kV}(-1) \hookrightarrow \pi_k^{-1}\bar{V}_{k-1} \xrightarrow{(\pi_k)^{-1}(\pi_{k-1})_*} \pi_k^{-1}\mathcal{O}_{\bar{P}_{k-1}V}(-1)$$

yields an effective divisor F_k. There is a canonical divisor on $\bar{P}_kV$ given by

$$\bar{P}_k^{\text{sing}}V = \bigcup_{2 \leq j \leq k} \pi_{j,k}^{-1}(F_j) \subset \bar{P}_kV.$$

Finally, let $\bar{P}_k^{\text{reg}}V = \bar{P}_kV \backslash \bar{P}_k^{\text{sing}}V$ and $\mathcal{O}_{\bar{P}_kV}(-1)^{\text{reg}} = \Big(\mathcal{O}_{\bar{P}_kV}(-1)|_{\bar{P}_k^{\text{reg}}V}\Big) \backslash \bar{P}_kV$, where the last $\bar{P}_kV$ denotes the zero section.

Proposition 2.1 *Let $(\bar{X}, D, \bar{V})$ be a log-directed manifold. If $\bar{V} \subset \bar{W} \subset \bar{T}X$ are holomorphic subbundles, then we have natural inclusions of submanifolds*

$$\bar{P}_k V \subset \bar{P}_k W \subset \bar{P}_k X.$$

□

Proposition 2.2 *Let $(\bar{X}, D, \bar{V})$ be a log-directed manifold. With the notations of Proposition 1.2 one has, over U, bundle isomorphisms*

$$K_k : \bar{P}_k V \to K^{-1}(\bar{P}_k(\mathbf{C}^r \setminus E)),$$

$$K'_{[k-1]} : \mathcal{O}_{\bar{P}_k V}(-1) \to K^{-1}(\mathcal{O}_{\bar{P}_k(\mathbf{C}^r \setminus E)}(-1)).$$

□

3 Log-directed jet differentials

3.1 Demailly-Semple jet bundles and jet differentials

In this subsection we recall some basic results of Demailly ([1]) for (usual) Demailly-Semple jet bundles, which we want to generalize to logarithmic Demailly-Semple jet bundles in the next subsections.

Let (X, V) be a directed manifold. Let

$$G_k = J_k \mathbf{C}_0^{\mathrm{reg}} = \{t \to \phi(t) = \sum_{i=1}^{k} a_i t^i \ , \ \ a_1 \in \mathbf{C}^*, \ a_i \in \mathbf{C}, \ i \geq 2\}$$

be the *group of reparametrizations.* Elements $\phi \in G_k$ act on $J_k V$ by

$$j_k(f) \to j_k(f \circ \phi).$$

Every nonconstant germ $f : (\mathbf{C}, 0) \to X$ tangent to V lifts to a unique germ $f_{[k]} : (\mathbf{C}, 0) \to P_k V$ tangent to V_k. It can be defined inductively to be the projectivization of $f'_{[k-1]} : (\mathbf{C}, 0) \to V_{k-1}$. Like that we also obtain a germ

$$f'_{[k-1]} : (\mathbf{C}, 0) \to \mathcal{O}_{\bar{P}_k V}(-1)$$

for the defining lift of $f_{[k]}$.

The *bundle of directed invariant jet differentials of order k and degree m*, denoted by $E_{k,m}V^*$, is defined as follows: $E_{k,m}V^*$ is the set of functions on $J_k V$ which satisfy

$$Q(j_k(f \circ \phi)) = \phi'(0)^m Q(j_k(f)) \ \ \forall \, j_k(f) \in J_k V^{\mathrm{reg}} \text{ and } \phi \in G_k. \tag{3.3}$$

Theorem 3.1 (Demailly, [1]) *Let (X,V) be a directed manifold.*
a) The maps

$$\tilde{\alpha}_k : J_kV^{\mathrm{reg}} \to \mathcal{O}_{P_kV}(-1)^{\mathrm{reg}} , \ \ j_k(f) \to f'_{[k-1]}(0),$$

$$\alpha_k : J_kV^{\mathrm{reg}} \to P_kV^{\mathrm{reg}} , \ \ j_k(f) \to f_{[k]}(0)$$

are holomorphic and surjective.
b) If $\phi \in G_k$ is a reparametrization, one has

$$(f \circ \phi)'_{[k-1]}(0) = f'_{[k-1]}(0) \cdot \phi'(0),$$

$$(f \circ \phi)_{[k]}(0) = f_{[k]}(0).$$

c) The quotient J_kV^{reg}/G_k of J_kV^{reg} by G_k has the structure of a locally trivial fiber bundle over X, and the map

$$\alpha_k/G_k \ : J_kV^{\mathrm{reg}}/G_k \to P_kV$$

is a holomorphic embedding which identifies J_kV^{reg}/G_k with P_kV^{reg}.
d) The direct image sheaf

$$(\pi_{0,k})_*\mathcal{O}_{P_kV}(m) \simeq \mathcal{O}(E_{k,m}V^*)$$

can be identified with the sheaf of holomorphic sections of $E_{k,m}V^*$.

3.2 Local trivializations

Proposition 3.2 *Let $z_1, ..., z_r$ be the standard coordinates of $\mathbf{C}^r$, let $a \leq r$, let $E = \{z_1...z_a = 0\}$ and $Q = (\overbrace{1, ..., 1}^{a}, \overbrace{0, ..., 0}^{r-a}) \in \mathbf{C}^r$. Then for the log-manifold $(\mathbf{C}^r, E)$ there exists an isomorphism*

$$\mathcal{O}_{\bar{P}_k(\mathbf{C}^r\setminus E)}(-1) \to \mathcal{O}_{P_k\mathbf{C}^r}(-1)_Q \times \mathbf{C}^r \tag{3.4}$$

such that the diagram

$$\begin{array}{ccc} \bar{J}_k(\mathbf{C}^r \setminus E)^{\mathrm{reg}}|_{\mathbf{C}^r\setminus E} & \to & J_k(\mathbf{C}^r)^{\mathrm{reg}}_Q \times (\mathbf{C}^r \setminus E) \\ \downarrow \alpha_k & & \downarrow \alpha_k \\ \mathcal{O}_{\bar{P}_k(\mathbf{C}^r\setminus E)}(-1)^{\mathrm{reg}}|_{\mathbf{C}^r\setminus E} & \to & \mathcal{O}_{P_k(\mathbf{C}^r)}(-1)^{\mathrm{reg}}_Q \times (\mathbf{C}^r \setminus E) \end{array} \tag{3.5}$$

commutes, where the upper isomorphism is induced by the coordinates of equation (1.8), that is, by the trivialization of $\bar{J}_k(\mathbf{C}^r \setminus E)$ by the forms $\omega^1 = \frac{dz_1}{z_1}, ..., \omega^a = \frac{dz_a}{z_a}$, $\omega^{a+1} = dz_{a+1}, ..., \omega^n = dz_n$. □

Important Remark 1: The local isomorphisms induced by the coordinates of equation (1.8) are fiber bundle isomorphisms, however, they are *not* induced by (directed) morphisms. (Even more, a lift of a germ of a curve is *not* mapped to a lift of a curve, any more. This fact is even necessary, because the logarithmic bundles on the left hand side can not be generated by lifts of germs of curves, the products on the right hand side, however, can, once a non- logarithmic local coordinate (trivialization) is chosen.) As a result, these local isomorphisms have a priori no kind of functoriality, and every compatibility which we will need has to be proved explicitly.

Proposition 3.3 *Let* $(\bar{X}, D, \bar{V})$ *be a log-directed manifold. With the notations of Proposition 1.2, and* $P = (\overbrace{1, ..., 1}^{l}, \overbrace{0, ..., 0}^{n-l}) \in U$ *(where* $D = \{z_1 ... z_l = 0\}$*), we have:*
a) The trivializations of equation (3.5) and Propositions 1.2 and 2.2 induce trivializations

$$\begin{array}{ccc} K^{-1}(\bar{J}_k(\mathbf{C}^r \setminus E))|_U & \rightarrow & (K^{-1}(J_k\mathbf{C}^r))_P \times U \\ \nwarrow & & \nwarrow \\ \bar{J}_k V|_U & \rightarrow & J_k V_P \times U \end{array} \tag{3.6}$$

and

$$\begin{array}{ccc} K^{-1}(\mathcal{O}_{\bar{P}_k(\mathbf{C}^r \setminus E)}(-1))|_U & \rightarrow & (K^{-1}(\mathcal{O}_{P_k\mathbf{C}^r}(-1)))_P \times U \\ \nwarrow & & \nwarrow \\ \mathcal{O}_{\bar{P}_k V}(-1)|_U & \rightarrow & \mathcal{O}_{P_k V}(-1)_P \times U \end{array} \tag{3.7}$$

b) Moreover, outside the divisor D, this induces the following cubic diagram

$$\begin{array}{ccccccc} K^{-1}(\bar{J}_k(\mathbf{C}^r \setminus E)^{\mathrm{reg}})|_{U \setminus D} & & \rightarrow & & (K^{-1}(J_k\mathbf{C}^r)^{\mathrm{reg}})_P \times (U \setminus D) & & \\ & \nwarrow & & & & \nwarrow & \\ & & \bar{J}_k V^{\mathrm{reg}}|_{U \setminus D} & & \rightarrow & & J_k V_P^{\mathrm{reg}} \times (U \setminus D) \\ \downarrow \tilde{\alpha}_k & & & & \downarrow \tilde{\alpha}_k & & \\ & & \downarrow \tilde{\alpha}_k & & & & \downarrow \tilde{\alpha}_k \\ K^{-1}(\mathcal{O}_{\bar{P}_k(\mathbf{C}^r \setminus E)}(-1)^{\mathrm{reg}})|_{U \setminus D} & & \rightarrow & & (K^{-1}(\mathcal{O}_{P_k\mathbf{C}^r}(-1))^{\mathrm{reg}})_P \times (U \setminus D) & & \\ & \nwarrow & & & & \nwarrow & \\ & & \mathcal{O}_{\bar{P}_k V}(-1)^{\mathrm{reg}}|_{U \setminus D} & & \rightarrow & & \mathcal{O}_{P_k V}(-1)_P^{\mathrm{reg}} \times (U \setminus D) \end{array} \tag{3.8}$$

c) By combining with the canonical line bundle projections we get the same diagrams with $\bar{P}_k(\mathbf{C}^r \setminus E)$, $\bar{P}_k V$ and α_k instead of $\mathcal{O}_{\bar{P}_k(\mathbf{C}^r\setminus E)}(-1)$, $\mathcal{O}_{\bar{P}_kV}(-1)$ and $\tilde{\alpha}_k$. □

Important Remark 2: For all local isomorphisms given by the horizontal left-to-right arrows in the above diagrams Remark 1 (see above) also holds. However, the local isomorphisms induced by the log-projection K are functorial.

3.3 Log-directed jets and logarithmic Demailly-Semple jets

In this subsection we want to extend Theorem 3.1, parts a), b) and c) to the log-directed case.

Let us start with part a) of this theorem. By Theorem 3.1, a), the map $\tilde{\alpha}_k$ is defined outside D:

$$\tilde{\alpha}_k : \bar{J}_k V^{\mathrm{reg}}|_{\bar{X}\setminus D} \to \mathcal{O}_{\bar{P}_kV}(-1)^{\mathrm{reg}}|_{\bar{X}\setminus D}. \tag{3.9}$$

Let $x \in D$. By Proposition 3.3 there exists a neighborhood U of x such that:

$$\begin{array}{ccc} \bar{J}_k V^{\mathrm{reg}}|_{U\setminus D} & \rightarrow & J_k V_P^{\mathrm{reg}} \times (U\setminus D) \\ \downarrow \tilde{\alpha}_k & & \downarrow \tilde{\alpha}_k \\ \mathcal{O}_{\bar{P}_kV}(-1)^{\mathrm{reg}}|_{U\setminus D} & \rightarrow & \mathcal{O}_{P_kV}(-1)_P^{\mathrm{reg}} \times (U\setminus D) \end{array} \tag{3.10}$$

Here the horizontal arrows are isomorphisms, and $\tilde{\alpha}_k$ is clearly extendable over D to a surjective holomorphic map on the right hand side. So it is also extendable to a surjective holomorphic map over U on the left hand side. Since $x \in D$ was arbitrary, and since by equation (3.9) the extension of $\tilde{\alpha}_k$ is unique if it exists, we obtain a well defined surjective holomorphic map

$$\tilde{\alpha}_k : \bar{J}_k V^{\mathrm{reg}} \to \mathcal{O}_{\bar{P}_kV}(-1)^{\mathrm{reg}}. \tag{3.11}$$

By combining with the canonical line bundle projections we get in the same way a surjective holomorphic map

$$\alpha_k : \bar{J}_k V^{\mathrm{reg}} \to \bar{P}_k V^{\mathrm{reg}}. \tag{3.12}$$

which extends the corresponding map α_k as defined in Theorem 3.1 from $\bar{X} \setminus D$ to $\bar{X}$.

Let us now deal with part b) of Theorem 3.1. If $\phi \in G_k$ is a reparametrization, one has on $\bar{J}_k V^{\mathrm{reg}}|_{\bar{X}\setminus D}$ by this theorem:

$$\tilde{\alpha}_k \circ \phi = \tilde{\alpha}_k \cdot \phi'(0)\ , \quad \alpha_k \circ \phi = \alpha_k, \tag{3.13}$$

where in the first equation the multiplication $\tilde{\alpha_k} \cdot \phi'(0)$ denotes the multiplication with scalars in the line bundle $\mathcal{O}_{\bar{P}_kV}(-1)^{\mathrm{reg}}|_{\bar{X}\setminus D}$.

Lemma 3.4 *For any $x \in D$, there exists a neighborhood U of x in $\bar{X}$ such that*

$$\begin{array}{ccc} \bar{J}_k V|_{U\setminus D} & \rightarrow & J_k V_P \times (U \setminus D) \\ \downarrow \circ\phi & & \downarrow \circ\phi \\ \bar{J}_k V|_{U\setminus D} & \rightarrow & J_k V_P \times (U \setminus D) \end{array} , \qquad (3.14)$$

where $\circ\phi$ denotes the operation $j_k(f) \rightarrow j_k(f \circ \phi)$. □

By a similar argument as in part a), the map $\circ\phi$ extends to a holomorphic map from $\bar{J}_k V$ to itself. Moreover, the same is true for the map $\circ\phi^{-1}$, and the equations $\phi \circ \phi^{-1} = \phi^{-1} \circ \phi = \mathrm{id}$ extend from $\bar{J}_k V|_{\bar{X}\setminus D}$ to $\bar{J}_k V$ by the identity theorem. Hence, the extended map $\circ\phi$ is actually a holomorphic automorphism on $\bar{J}_k V$. Finally, equation (3.13) extends from $\bar{J}_k V^{\mathrm{reg}}|_{\bar{X}\setminus D}$ to $\bar{J}_k V^{\mathrm{reg}}$ again by the identity theorem.

Let us deal now deal with part c) of Theorem 3.1. We have seen already that the elements $\phi \in G_k$ yield holomorphic automorphisms $\circ\phi$ on $\bar{J}_k V$ over $\bar{X}$, which extend the reparametrization operation of ϕ on $\bar{J}_k V|_{\bar{X}\setminus D}$. From equation (3.14) it is also clear that $\circ\phi$ maps $\bar{J}_k V^{\mathrm{reg}}$ onto itself, since this is true for $\bar{J}_k V_P^{\mathrm{reg}}$. So the quotients $\bar{J}_k V / G_k$ and $\bar{J}_k V^{\mathrm{reg}} / G_k$ are well defined (as sets).

By the diagrams of equations (3.13) and (3.14), we obtain from part c) of Proposition 3.3:

$$\begin{array}{ccc} \bar{J}_k V^{\mathrm{reg}}/G_k|_{U\setminus D} & \rightarrow & (J_k V^{\mathrm{reg}}/G_k)_P \times (U \setminus D) \\ \downarrow \alpha_k/G_k & & \downarrow \alpha_k/G_k \\ \bar{P}_k V^{\mathrm{reg}}|_{U\setminus D} & \rightarrow & P_k V_P^{\mathrm{reg}} \times (U \setminus D) \end{array} \qquad (3.15)$$

By Demailly ([1]), the vertical arrows in this diagram are isomorphisms. By a similar argument like in part b), one obtains from that a holomorphic isomorphism

$$\alpha_k/G_k : \bar{J}_k V^{\mathrm{reg}}/G_k \rightarrow \bar{P}_k V^{\mathrm{reg}} \qquad (3.16)$$

over $\bar{X}$ which extends the fiber bundle isomorphism α_k/G_k in Theorem 3.1, c). Especially this isomorphism gives $\bar{J}_k V^{\mathrm{reg}}/G_k$ the structure of a holomorphic fibre bundle over $\bar{X}$.

3.4 Characterization of log-directed jet differentials

In this subsection we generalize part d) of Theorem 3.1. More precisely we prove:

Proposition 3.5 *A holomorphic (respectively meromorphic) function Q on $\bar{J}_k V|_U$ for some open subset $U \subset \bar{X}$ which satisfies*

$$Q(j_k(f \circ \phi)) = \phi'(0)^m Q(j_k(f)) \ \ \forall\, j_k(f) \in J_k V^{\mathrm{reg}} \text{ and } \phi \in G_k$$

over some open subset of U' of $U \setminus D$ defines a holomorphic (respectively meromorphic) section of $\mathcal{O}_{\bar{P}_k V}(m)$ over U, and vice versa. □

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