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# ON THE HYPERBOLICITY OF THE COMPLEMENTS OF CURVES IN ALGEBRAIC SURFACES: THE THREE-COMPONENT CASE

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**1. Introduction.** In complex analysis, hyperbolic manifolds have been studied extensively, with close relationships to other areas (cf., e.g., [20]). Hyperbolic manifolds are generalizations of hyperbolic Riemann surfaces to higher dimensions. Despite the fact that the general theory of hyperbolic manifolds is well developed, only very few classes of hyperbolic manifolds are known. But one could hope that "most" of the pseudoconvex quasi-projective varieties are in fact hyperbolic, provided that the degrees involved are high enough. In particular, it is believed that, e.g., the complements of most hypersurfaces in $\mathbb{P}_n$ are hyperbolic, if only their degree is at least $2n+1$. More precisely, according to Kobayashi [18] and later Zaidenberg [30], one has the following.

CONJECTURE. *Let $\mathscr{C}(d_1, \ldots, d_k)$ be the space of k-tuples of hypersurfaces $C = (C_1, \ldots, C_k)$ in $\mathbb{P}_n$, where $\deg(C_i) = d_i$. Then, for all $(d_1, \ldots, d_k)$ with $\sum_{i=1}^k d_i =: d \geqslant 2n+1$, the set $\mathscr{H}(d_1, \ldots, d_k) = \{C \in \mathscr{C}(d_1, \ldots, d_k)\colon \mathbb{P}_n \setminus \bigcup_{i=1}^k C_i$ is complete hyperbolic and hyperbolically embedded$\}$ contains the complement of a proper algebraic subset of $\mathscr{C}(d_1, \ldots, d_k)$.*

In this paper we shall restrict ourselves to the two-dimensional case. However, we consider also more general quasi-projective complex surfaces than the complements of curves in the projective plane.

Concerning the above conjecture, the following is known: it seems that the conjecture is the more difficult the smaller $k$ is. Other than in the case of five lines ($\mathscr{C}(1,1,1,1,1)$), the conjecture was proved by M. Green in [15] in the case of a curve $C$ consisting of one quadric and three lines ($\mathscr{C}(2,1,1,1)$). Furthermore, it was shown for $\mathscr{C}(d_1, \ldots, d_k)$, whenever $k \geqslant 5$, by Babets in [3]. A result which went much further was given by Eremenko and Sodin in [9], where they proved a second main theorem of value distribution theory in the situation $k \geqslant 5$. Green

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proved in [14] that, for any hypersurface $C$ consisting of at least four components in $\mathbb{P}_2$, any entire curve $f: \mathbb{C} \to \mathbb{P}_2\backslash C$ is algebraically degenerate. Knowing this, it follows immediately that, for generic configurations, any such algebraically degenerate map is constant, and hence the conjecture is true for any family $\mathscr{C}(d_1, \ldots, d_k)$ with $k \geqslant 4$ (cf. [8]). (The degeneracy locus of the Kobayashi pseudometric was studied by Adachi and Suzuki in [1], [2].)

In our paper [8] we gave a proof of the conjecture for three quadrics ($\mathscr{C}(2, 2, 2)$), based on methods from value distribution theory. The three-quadric case had been previously studied by Grauert (in [11]), who used differential geometry. However, certain technical problems still exist with this approach. For $\mathscr{C}(2, 2, 1)$, i.e., two quadrics and a line, we proved with similar methods the existence of an open set in the space of all such configurations, which contains a quasi-projective set of codimension one, where the conjecture is true.

The paper contains two main results. The first is Theorem 6.1. It states that the conjecture is true for almost all three-component cases, namely for $\mathscr{C}(d_1, d_2, d_3)$ with $d_1, d_2, d_3 \geqslant 2$ and at least one $d_i \geqslant 3$. Together with our result for three quadrics (which, by the way, occur on the borderline of the method used in this paper), this means that the conjecture is true for three components whenever none of them is a line. We finally remark that we get a weaker conclusion also for $\mathscr{C}(d_1, d_2, d_3)$ where, up to enumeration, $d_1 = 1$, $d_2 \geqslant 3$, $d_3 \geqslant 4$: namely, we show that any holomorphic map $f: \mathbb{C} \to X$ is algebraically degenerated, i.e., $f(\mathbb{C})$ is contained in a proper algebraic subset of $X$.

The other main result is Theorem 6.2. We consider a smooth surface $\overline{X}$ in $\mathbb{P}_3$ of degree at least five for which every curve on $\overline{X}$ is the complete intersection with another hypersurface. Surfaces of this kind are much more general than $\mathbb{P}_2$—by the Noether-Lefschetz theorem (cf. [6]) the "generic" surface in $\mathbb{P}_3$ of any given degree at least four has this property. ("Generic" here indicates the complement of a countable union of proper varieties.) Let $C$ be a curve on $\overline{X}$ consisting of three smooth components intersecting transversally. From our assumptions we know that $C$ is a complete intersection of $\overline{X}$ and a hypersurface $B$. We assume that the degree of $B$ is at least five. Now Theorem 6.2 states the hyperbolicity of any such $X = \overline{X}\backslash C$. Moreover $X$ is complete hyperbolic and hyperbolically embedded.

Our method of proof is the following: We heavily use a theorem due to S. Lu (cf. [22]). It states that for a certain class of differentials $\sigma$, which may have logarithmic poles along the curve $C$, and any holomorphic map $f: \mathbb{C} \to \mathbb{P}_2\backslash C$, the pullback $f^*(\sigma)$ vanishes identically. This can be interpreted as algebraic degeneracy of the tangential map corresponding to $f: \mathbb{C} \to X$. Our aim is to show algebraic degeneracy of the map $f$ itself.

The paper is organized as follows: In Section 2 we collect, for the convenience of the reader, some basics from value distribution theory. (Readers who are familar with these may skip this section.) In Section 3 we fix the notation and quote some theorems which are needed in the following proof, especially Lu's theorem. Furthermore, we examine more closely the spaces of sections which are used in

Lu's theorem and get sections with special zero sets. The essential step of our paper is the proof of Theorem 4.2 in Section 4. It states the algebraic degeneracy of holomorphic maps $f: \mathbb{C} \to X$, if $\mathrm{Pic}(\overline{X}) = \mathbb{Z}$, and under assumptions on the determinant bundle and the Chern numbers of the logarithmic cotangent bundle on $\overline{X}$ with respect to $C$. The proof uses value distribution theory and the existence of the special sections which were constructed in Section 3. In Section 5 we compute the Chern numbers and the determinant bundle in the situation where $\overline{X}$ is a complete intersection (Theorem 5.3). We apply this to $\overline{X} = \mathbb{P}_2$, and to hypersurfaces in $\mathbb{P}_3$ using the Noether-Lefschetz theorem (Theorem 5.4). Finally, in Section 6, we apply Theorem 5.4 and get Theorem 6.1, using an argument like in our paper [8] to prove the nonexistence of algebraic entire curves in generic complements. Furthermore, we apply Theorem 5.4 using results of Xu [29] and Clemens [7] to get Theorem 6.2.

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**2. Some tools from value distribution theory.** In this section we fix some notations and quote some facts from value distribution theory. We give references but do not trace these facts back to the original papers.

We define the characteristic function and the counting function, and give some formulas for these.

Let $\|z\|^2 = \sum_{j=0}^{n} |z_j|^2$, where $(z_0, \dots, z_n) \in \mathbb{C}^{n+1}$, let $\Delta_t = \{\xi \in \mathbb{C}: |\xi| < t\}$, and let $d^c = (i/4\pi)(\bar\partial - \partial)$. Let $r_0$ be a fixed positive number and let $r \geqslant r_0$. Let $f: \mathbb{C} \to \mathbb{P}_n$ be an entire holomorphic curve; i.e., $f$ can be written as $f = [f_0 : \cdots : f_n]$ with holomorphic functions $f_j: \mathbb{C} \to \mathbb{C}$, $j = 0, \dots, n$ without common zeroes. Then the *characteristic function* $T(f, r)$ is defined as

$$T(f, r) = \int_{r_0}^{r} \frac{dt}{t} \int_{\Delta_t} dd^c \log \|f\|^2.$$

Let furthermore $D = V(P)$ be a divisor in $\mathbb{P}_n$, given by a homogeneous polynomial $P$. Assume $f(\mathbb{C}) \not\subset \mathrm{support}(D)$. Let $n_f(D, t)$ denote the number of zeroes of $P \circ f$ inside $\Delta_t$ (counted with multiplicities). Then we define the *counting function* as

$$N_f(D, r) = \int_{r_0}^{r} n_f(D, t) \frac{dt}{t}.$$

The Stokes theorem and transformation to polar coordinates imply (cf. [28]):

$$T(f, r) = \frac{1}{4\pi} \int_0^{2\pi} \log \|f\|^2 (re^{i\vartheta})\, d\vartheta + O(1). \tag{1}$$

The characteristic function as defined by Nevanlinna for a holomorphic function $f: \mathbb{C} \to \mathbb{C}$ is

$$T_0(f, r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\vartheta})| \, d\vartheta.$$

For the associated map $[1:f]: \mathbb{C} \to \mathbb{P}_1$, one has

$$T_0(f, r) = T([1:f], r) + O(1) \tag{2}$$

(cf. [16])

By abuse of notation we will, from now on, for a function $f: \mathbb{C} \to \mathbb{C}$, write $T(f, r)$ instead of $T_0(f, r)$. Furthermore, we sometimes use $N(f, r)$ instead of $N_f(z_1 = 0, r)$.

We state some elementary properties of the characteristic function.

LEMMA 2.1. *Let $f, g, f_j: \mathbb{C} \to \mathbb{C}$ be entire holomorphic functions for $j = 0, \ldots, n$. Then*

(a) $T(f \cdot g, r) \leqslant T(f, r) + T(g, r) + O(1)$;
(b) $T([f_0 : \cdots : f_n], r) \leqslant \sum_{j=0}^{n} T(f_j, r) + O(1)$;
(c) $T(f + g, r) \leqslant T(f, r) + T(g, r) + O(1)$.

Later we will use the concept of finite order.

*Definition 2.2.* Let $s(r)$ be a positive, monotonically increasing function defined for $r \geqslant r_0$. If

$$\overline{\lim_{r \to \infty}} \frac{\log s(r)}{\log r} = \lambda$$

then $s(r)$ is said to be of order $\lambda$. For an entire holomorphic curve $f: \mathbb{C} \to \mathbb{P}_n$ or an entire function $f: \mathbb{C} \to \mathbb{C}$, we say that $f$ is of order $\lambda$, if $T(f, r)$ is.

For holomorphic maps to $\mathbb{P}_1$ whose characteristic function only grows like $\log r$, we have the following characterization (cf. [16]).

LEMMA 2.3. *Let $f = [f_0 : f_1]: \mathbb{C} \to \mathbb{P}_1$ be holomorphic. Then $T(f, r) = O(\log r)$ if and only if the meromorphic function $f_0/f_1$ is equal to a quotient of two polynomials.*

We need the following

LEMMA 2.4. *Assume that $f: \mathbb{C} \to \mathbb{P}_n$ is an entire holomorphic curve which misses the divisors $\{z_j = 0\}$ for $j = 0, \ldots, n$ (i.e., the coordinate hyperplanes of $\mathbb{P}_n$). Assume that $f$ has order at most $\lambda$. Then $f$ can be written as $f = [1 : f_1 : \cdots : f_n]$ with $f_j(\xi) = e^{P_j(\xi)}$, where the $P_j(\xi)$ are polynomials in $\xi$ of degree $d_j \leqslant \lambda$.*

*Proof.* We write $f = [1 : f_1 : \cdots : f_n]$ with holomorphic $f_j: \mathbb{C} \to \mathbb{C} \backslash \{0\}$. Now we get with equation (1)

$$T(f_j, r) = T([1 : f_j], r) + O(1) \leqslant T(f, r) + O(1),$$

and hence the $f_j$ are nonvanishing holomorphic functions of order at most $\lambda$. This means that

$$\limsup_{r\to\infty} \frac{T(f_j, r)}{r^{\lambda+\varepsilon}} = 0$$

for any $\varepsilon > 0$ From this equation our assertion follows from the Weierstraß theorem as it is stated in [16]. □

In the following paragraphs, we state the First and Second Main Theorem of value distribution theory, due to Cartan and Ahlfors, which relate the characteristic function and the counting function (cf. [27])

Let $f: \mathbb{C} \to \mathbb{P}_n$ be an entire holomorphic curve, and let $D$ be a divisor in $\mathbb{P}_n$ of degree $d$, such that $f(\mathbb{C}) \not\subset \text{support}(D)$

FIRST MAIN THEOREM

$$N_f(D, r) \leqslant d \cdot T(f, r) + O(1).$$

Assume now that $f(\mathbb{C})$ is not contained in any hyperplane in $\mathbb{P}_n$, and let $H_1, \ldots, H_q$ be distinct hyperplanes in general position.

SECOND MAIN THEOREM

$$(q - n - 1)T(f, r) \leqslant \sum_{j=0}^{q} N_f(H_j, r) + S(r)$$

*where* $S(r) \leqslant O(\log(rT(f, r)))$ *for all* $r \geqslant r_0$ *except for a set of finite Lebesque measure. If* $f$ *is of finite order, then* $S(r) \leqslant O(\log r)$ *for all* $r \geqslant r_0$

**3. Setup and basic methods.** We denote by $\bar{X}$ a nonsingular projective surface and by $C$ a curve in $\bar{X}$ whose irreducible components are smooth and intersect each other only in normal crossings. Let $X = \bar{X}\backslash C$.

We denote by $\Omega^1_{\bar{X}}(\log C)$ the bundle of holomorphic one-forms of $\bar{X}$ with logarithmic poles along $C$ introduced by Deligne (cf. [22], page 310). Let $E$ be its dual. Then we define the projectivized logarithmic tangent bundle $p: \mathbb{P}(E) \to \bar{X}$ over $\bar{X}$ to be the projectivized bundle whose fibers correspond to the one-dimensional subspaces of the fibers of $E$. Furthermore, let $\mathcal{O}_{\mathbb{P}(E)}(-1)$ be the sheaf associated to the tautological line bundle on $\mathbb{P}(E)$, for which we have the canonical isomorphism between the total space of $\mathcal{O}_{\mathbb{P}(E)}(-1)\backslash\{\text{zerosection}\}$ and the total space of $E\backslash\{\text{zerosection}\}$.

Let $D$ be a divisor on $\bar{X}$ According to a theorem of Kobayashi-Ochiai (cf [19]), the cohomology, in particular the holomorphic sections, of a symmetric power of $E^*$ tensor multiplied with the bundle $[-D]$, corresponds to the cohomology of the $m$th power of the dual of the tautological line bundle on $\mathbb{P}(E)$, tensor multiplied with the pullback of $[-D]$:

$$H^0(\overline{X}, S^m(E^*) \otimes [-D]) \simeq H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m) \otimes p^*[-D]),$$

and in particular

$$H^0(\overline{X}, S^m(E^*)) \simeq H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m)).$$

Let $f: \mathbb{C} \to X$ be a holomorphic map. Denote by

$$(f, f'): T(\mathbb{C}) \to T(X)$$

the induced map from $T(\mathbb{C})$ to the holomorphic tangent bundle $T(X)$, which gives rise to a meromorphic map

$$F: \mathbb{C} \to \mathbb{P}(T(X))$$

from $\mathbb{C}$ to $\mathbb{P}(T(X))$. Since the domain is of dimension one, points of indeterminacy can be eliminated, more precisely the map $F$ extends holomorphically into the points $\xi \in \mathbb{C}$ where $f'(\xi) = 0$. We denote the extended holomorphic map on $\mathbb{C}$ again by $F$. Since the restriction of $E$ to $X$ is isomorphic to the holomorphic tangent bundle of $X$, any map

$$f: \mathbb{C} \to X$$

has a unique holomorphic lift $F: \mathbb{C} \to \mathbb{P}(E)$.

The following theorem imposes restrictions to such lifts $F$. It is a special case of Theorem 2 of Lu in [22]. (Where $\Xi = P$ in Lu's notation, which is our $\mathbb{P}(E)$, his ample line bundle $H$ is our $[D]$.) It actually follows from Proposition 4.1 there.

THEOREM 3.1 (Lu). *Assume that the divisor $D$ is ample and that there exists a nontrivial holomorphic section*

$$0 \neq \sigma \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m) \otimes p^*[-D]).$$

*Then, for any nonconstant holomorphic map $f: \mathbb{C} \to X$, the holomorphic lift $F: \mathbb{C} \to \mathbb{P}(E)$ has image in the zero set of $\sigma$.*

In order to apply Lu's theorem in a given situation, it is important to guarantee the existence of suitable sections.

Let $(\overline{X}, C)$ be given as above. The logarithmic Chern classes $\bar{c}_j(X)$ are by definition the Chern classes of the logarithmic tangent bundle $E$:

$$\bar{c}_j(X) = c_j(E) = c_j(X, (\Omega^1_{\overline{X}}(\log C))^*).$$

Now the existence of suitable sections is guaranteed by the following theorem of Bogomolov (cf. [5] and also Lu [22], Proof of Proposition 3.1 and localization to the divisor $D$).

THEOREM 3.2 (Bogomolov) *Let $D$ be a divisior on $\overline{X}$, $D$ effective (i.e., $D \geqslant 0$). Assume that*

$$\bar{c}_1^2(X) - \bar{c}_2(X) > 0,$$

*and that*

$$\det(E^*)$$

*is effective. Then there exists a positive constant $A$ and $m_0, n_0 \in \mathbb{N}$, such that*

$$A \cdot m^3 \leqslant h^0(\overline{X}, S^{mn_0}(E^*) \otimes [-D])$$

*for all $m \geqslant m_0$.*

We have the following nonexistence statement, which is a consequence of the logarithmic version of the Bogomolov's lemma due to Sakai (cf. [26]). It will also become important for the following proofs.

LEMMA 3.3. *Assume that the divisor $D$ is ample. Then the following group vanishes:*

$$H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1) \otimes p^*[-D]) = \{0\}.$$

*In particular, there is no logarithmic one-form on $\overline{X}$ which vanishes on $D$.*

*Proof.* The existence of a nontrivial section $s \in H^0(\overline{X}, \Omega^1_{\overline{X}}(\log C) \otimes [-D])$ implies that the invertible sheaf $\mathscr{L} := [D]$ can be realized as mapped into $\Omega^1_{\overline{X}}(\log C)$. According to [26], (7.5), this implies that the $\mathscr{L}$-dimension of $\overline{X}$ equals one, which is clearly impossible since $[D]$ is ample. □

Next we deal with divisors in $\mathbb{P}(E)$ which project down to all of $\overline{X}$.

*Definition 3.4.* Consider the projection $p: \mathbb{P}(E) \to \overline{X}$. We call a divisor $Z \subset \mathbb{P}(E)$ horizontal if $p(Z) = \overline{X}$.

Those horizontal divisors which occur as parts of the zero sets $V(\sigma)$ of sections $0 \neq \sigma \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m) \otimes p^*[-D])$ will play an important role in the sequel. We study this relationship somewhat more closely.

LEMMA 3.5. *Given*

$$0 \neq \sigma \in H^0(\mathbb{P}(E), \mathcal{O}(m) \otimes p^*[-D]),$$

*there exist divisors* $E_j$, $j = 1, \ldots, l$ *on* $\overline{X}$, *and numbes* $a_j, n_j \in \mathbb{N}$ *such that* $[\sum a_j \cdot E_j - D] \geqslant 0$ *and sections* $s_j \in H^0(\mathbb{P}(E), \mathcal{O}(n_j) \otimes p^*[-E_j])$, $\tau \in H^0(\mathbb{P}(E), p^*[\sum a_j \cdot E_j - D])$ *such that* $\sigma = \tau \cdot \prod_{1 \leqslant j \leqslant l} s_j^{a_j}$ *with the following property: the zero sets of* $s_j$ *are precisely the irreducible horizontal components of* $V(\sigma)$.

*Proof.* Let $0 \neq \sigma \in H^0(\mathbb{P}(E), \mathcal{O}(m) \otimes p^*[-D])$ be a nontrivial section and $V(\sigma)$ its zero divisor. We denote by $S_j$; $j = 1, \ldots, l$ the irreducible horizontal components of $V(\sigma)$. Since $\mathrm{Pic}(\mathbb{P}(E)) = \mathrm{Pic}(\overline{X}) \oplus \mathbb{Z}$, we get $[S_j] = \mathcal{O}_{\mathbb{P}(E)}(n_j) \otimes p^*[-E_j]$ for certain (not necessarily effective) divisors $E_j$ on $\overline{X}$ and $n_j \geqslant 1$. The last fact follows by restricting the bundles $[S_j]$ to a generic fiber of $p$. Let $a_j$ be the multiplicities of $\sigma$ with respect to $S_j$, then in particular $a_1 n_1 + \cdots + a_l n_l = m$. (This fact follows again by restricting bundles and sections to a generic fiber of $p$.) Canonical sections of $[S_j]$ give rise to nontrivial sections $s_j \in H^0(\mathbb{P}(E), \mathcal{O}(n_j) \otimes p^*[-E_j])$, which vanish exactly on $S_j$. Thus $\tau := \sigma/(s_1^{a_1} \cdot \cdots \cdot s_l^{a_l})$ is a (holomorphic) section of $H^0(\mathbb{P}(E), p^*[\sum a_j \cdot E_j - D])$. In particular, $[\sum a_j \cdot E_j - D] \geqslant 0$. □

In order to control the horizontal divisors of a section of $H^0(\mathbb{P}(E), \mathcal{O}(m) \otimes p^*[-D])$, the number $m$ will be chosen minimal in the following sense.

For any $k \in \mathbb{N}$, we set

$$\mu_k := \inf\{m;\ h^0(\mathbb{P}(E), \mathcal{O}(m) \otimes p^*[-kD]) > 0\}.$$

and

$$\mu := \inf_{k \in \mathbb{N}} \{\mu_k\}.$$

LEMMA 3.6. *Assume that* $\mathrm{Pic}(\overline{X}) = \mathbb{Z}$ *and that* $[D]$ *is the ample generator of* $\mathrm{Pic}(\overline{X})$. *Furthermore, assume that*

$$\bar{c}_1^2(X) - \bar{c}_2(X) > 0,$$

*and that*

$$\det(E^*)$$

*is effective.*

*Then we have*

$$2 \leqslant \mu < \infty$$

*and if* $k_0 = \min\{k \in \mathbb{N}: \mu_k = \mu\}$, *there exists a nontrival section*

$$0 \neq \sigma \in H^0(\mathbb{P}(E), \mathcal{O}(\mu) \otimes p^*[-k_0 D])$$

*such that exactly one horizontal component of* $V(\sigma)$ *exists and has multiplicity one.*

*Proof.* Since some multiple of $D$ is very ample and hence linearly equivalent to an effective divisor, we have $\mu < \infty$ from Theorem 3.2. From Lemma 3.3 we then get $\mu \geqslant 2$.

Now take any section $0 \neq \sigma \in H^0(\mathbb{P}(E), \mathcal{O}(\mu) \otimes p^*[-k_0 D])$. We use Lemma 3.5. Since $[D]$ is a generator of $\mathrm{Pic}(\overline{X})$, there exist $b_j \in \mathbb{Z}$ such that $[E_j] = b_j \cdot [D]$. Since $[\sum a_j \cdot E_j - k_0 D] \geqslant 0$, we have $\sum a_j b_j \geqslant k_0$. Since all $a_j \geqslant 0$, there must be at least one $b_j > 0$, say $b_1 > 0$. Now $s_1 \in H^0(\mathbb{P}(E), \mathcal{O}(n_1) \otimes p^*[-b_1 \cdot D])$ is a non-trivial section. By definition of $\mu$ we have $n_1 \geqslant \mu$, which means $n_1 = \mu$, since $\sum a_j n_j = \mu$. So in terms of the notion of Lemma 3.5, $\sigma = \tau \cdot s_1$, i.e., $S$ contains only one horizontal component. This component has multiplicity one. □

**4. Algebraic degeneracy of entire curves.** Let $\overline{X}$ be a nonsingular (connected) projective surface.

*Definition 4.1.* Let $f: \mathbb{C} \to \overline{X}$ be a holomorphic map. We call $f$ *algebraically degenerate* if there exists an algebraic curve $A \subset \overline{X}$ such that $f(\mathbb{C})$ is contained in $A$.

The following theorem is our main result on algebraic degeneration.

THEOREM 4.2. *Let $C \subset \overline{X}$ be a curve consisting of three smooth components with normal crossings. Assume that:*

(i) $\mathrm{Pic}(\overline{X}) = \mathbb{Z}$.

(ii) *The logarithmic Chern numbers of $X = \overline{X} \backslash C$ satisfy the inequality*

$$\bar{c}_1^2(X) - \bar{c}_2(X) > 0$$

(iii) *The line bundle* $\det(E^*)$ *is effective, where* $E^* = \Omega_{\overline{X}}^1(\log C)$ *is the logarithmic cotangent bundle.*

*Then any holomorphic map $f: \mathbb{C} \to \overline{X} \backslash C$ of order at most two is algebraically degenerate.*

*Remark.* The theorem also holds without the assumption on the order of the map $f$, but since we are mostly interested in the hyperbolicity of the complement, we include this assumption, because it slightly simplifies the proof.

The rest of this section is devoted to the proof of this theorem. Let again $[D]$ be an ample generator of $\mathrm{Pic}(\overline{X})$. Let $k \in \mathbb{N}$ be a natural number such that $[kD]$ is very ample. Then by Theorem 3.2 there exists a symmetric differential $\omega \in H^0(\overline{X}, S^m(E^*) \otimes [-kD])$ which is not identically zero. By Theorem 3.1 we know that $f^*\omega \equiv 0$.

The proof now will work as follows: The three components of the curve $C$ give rise to a morphism $\Phi: \overline{X} \to \mathbb{P}_2$, which maps $C$ to the union of the three coordinate axes. In the first step of the proof we show that we can "push down" the symmetric differential $\omega$ by this morphism to some symmetric rational differential $\Omega$ on $\mathbb{P}_2$ and that we still have $(\Phi \circ f)^*(\Omega) \equiv 0$. Since $\Phi \circ f$ maps the complex plane to the complement of the three coordinate hyperplanes in $\mathbb{P}_2$, we will be able to interpret this, in the second step of the proof, as an equation for non-vanishing functions with coefficients which may have zeroes, but which grow of smaller order, only. In such a situation we then can apply value distribution theory.

*First step.* We first remark that the intersection number of any two curves $D_1$ and $D_2$ is positive (including self-intersection numbers). Let $[D]$ be the ample generator of $\mathrm{Pic}(\overline{X}) = \mathbb{Z}$. Now $[D_j] = a_j[D]$; $a_j \in \mathbb{Z}$, and $0 < D_j \cdot D = a_j D^2$ (cf. the easy implication of the Nakai criterion). Hence all $a_j$ are positive, and

$$D_1 \cdot D_2 = a_1 a_2 D^2 > 0.$$

We can find $a_j \in \mathbb{N}$; $j = 1, 2, 3$ such that $[a_1 C_1] = [a_2 C_2] = [a_3 C_3] = L$, since the divisors $C_j$ $j = 1, 2, 3$ are effective. Let $\sigma_j \in H^0(\overline{X}, L)$ be holomorphic sections which vanish exactly on $C_j$. Then

$$\Phi = [\sigma_1 : \sigma_2 : \sigma_3] : \overline{X} \to \mathbb{P}_2$$

defines a rational map, which is a morphism, since the three components do not pass through any point of $\overline{X}$.

LEMMA 4.3. *The morphism $\Phi$ is a branched covering.*

*Proof.* Since $C_2 \cdot C_3 > 0$, the fiber $\Phi^{-1}(1:0:0) = C_2 \cap C_3$ is nonempty. By assumption, $C_2 \cap C_3$ consists of at most finitely many points. Hence $\Phi$ is surjective and has discrete generic fibers. Finally $\Phi$ has no positive-dimensional fibers at all: from Stein factorization we would get a bimeromorphic map. Since there are no curves of negative self-intersection, no exceptional curves exist on $\overline{X}$ (cf. [4]). Hence there exist no positive-dimensional fibers of $\Phi$. □

Hence the morphism $\Phi$ is a finite branched covering of $\overline{X}$ over $\mathbb{P}_2$ with, let us say, $N$ sheets. Let $R$ be the ramification divisor of $\Phi$, $B = \Phi(R)$ the branching locus, and $R' = \Phi^{-1}(B)$. Then

$$\Phi\colon \overline{X}\backslash R' \to \mathbb{P}_2\backslash B$$

is an unbranched covering with $N$ sheets.

We now want to construct a meromorphic symmetric $mN$-form $\Omega$ defined on $\mathbb{P}_2\backslash B$ from the meromorphic symmetric $m$-form $\omega$ on $\overline{X}$: For any point $w^0 \in \mathbb{P}_2\backslash B$, there exists a neighborhood $U = U(w^0)$ of $w^0$ and $N$ holomorphic maps $a_i(w)$, $a_i\colon U \to \overline{X}\backslash R'$; $i = 1, \ldots, N$ such that $\Phi \circ a_i = \mathrm{id}_U$. By pulling back the symmetric $m$-form $\omega$ by means of these maps, we get $N$ meromorphic symmetric $m$-forms $(a_i)^*(\omega)(w)$ on $U$. Taking now the symmetric product of these $m$-forms, we get the symmetric $(Nm)$-form $\Omega$ on $U$:

$$\Omega(w) = \prod_{i=1}^{N} a_i^* \omega(w)$$

Let $M = Nm$ Defining $g = \Phi \circ f$, we then have the following.

LEMMA 4.4. *The form $\Omega$ extends to a rational symmetric $M$-form on $\mathbb{P}_2$, which we again denote by $\Omega$. We have $\Omega \not\equiv 0$, but $g^*\Omega = 0$.*

The first statement of this lemma is probably well known, and the second statement is considered to be obvious. But since we did not find a reference, we will include a proof of this lemma at the end of this section.

We proceed with the proof of Theorem 4.2. Denote the homogeneous coordinates of $\mathbb{P}_2$ by $w_0$, $w_1$, $w_2$ On $\mathbb{P}_2 \backslash V(w_0)$, we have inhomogeneous coordinates $\xi_1 = w_1/w_0$, $\xi_2 = w_2/w_0$. Hence on $\mathbb{P}_2 \backslash V(w_0)$ the symmetric $M$-form $\Omega$ can be written as

$$\Omega = \sum_{\mu=1}^{M} R_\mu(\xi_1, \xi_2)(d\xi_1)^\mu (d\xi_2)^{M-\mu} \tag{3}$$

where multiplication means the symmetric tensor product here, and the coefficients $R_\mu(\xi_1, \xi_2)$ are rational functions in $\xi_1$ and $\xi_2$.

Now $g: \mathbb{C} \to \mathbb{P}_2$ has values in the complement $\mathbb{P}_2 \backslash V(w_0 w_1 w_2)$ of the three-coordinate axes, and hence the functions $g_j = \xi_j \circ g$ are holomorphic and without zeroes Since $g^*\Omega \equiv 0$ on $\mathbb{C}$, equation (3) implies

$$\sum_{\mu=1}^{M} R_\mu(g_1(\eta), g_2(\eta))(g_1'(\eta))^\mu (g_2'(\eta))^{M-\mu} \equiv 0 \tag{4}$$

for all $\eta \in \mathbb{C}$. This equation still holds if we clear the denominators of the $R_\mu(\xi_1, \xi_2)$ simultaneously, so without loss of generality we may assume from now on that in equation (4) the $R_\mu(g_1(\eta), g_2(\eta))$ are polynomials in $g_1(\eta)$ and $g_2(\eta)$; i.e., we have

$$R_\mu(g_1(\eta), g_2(\eta)) = \sum_{j,k} a_{\mu jk}(g_1(\eta))^j (g_2(\eta))^k. \tag{5}$$

Under our assumptions, we are able to say more about the functions $g_i$; $i = 1$, 2. Since the holomorphic map $f: \mathbb{C} \to \overline{X} \backslash C$ was of finite order at most two, this is also true for $g = \Phi \circ f$ by Lemma 2.1, since the components of $g$ are polynomials in the components of $f$

Hence, by Lemma 2.4, we have

$$g_i(\eta) = \exp(p_i(\eta)) \tag{6}$$

where the $p_i(\eta)$; $i = 1, 2$ are polynomials in $\eta$ of degree at most two. Furthermore we may assume that both polynomials are nonconstant; otherwise, $g$ would be linearly degenerate and so $f$ would be algebraically degenerate, and the proof would be finished.

Replacing equation (6) and equation (5) in equation (4), we get

$$\sum_{\mu=1}^{M} \sum_{j,k} a_{\mu jk} \exp\{(\mu + j)p_1(\eta) + (M - \mu + k)p_2(\eta)\}(p_1'(\eta))^{\mu}(p_2'(\eta))^{M-\mu} \equiv 0. \quad (7)$$

If we still allow linear combinations of the above summands with constant coefficients $c_{\mu jk}$ in equation (7), we can pass to a subset $S$ of indices which occur in this equation and get a relation

$$\sum_{(\mu, j, k) \in S} c_{\mu jk} a_{\mu jk} \exp\{(\mu + j)p_1(\eta) + (M - \mu + k)p_2(\eta)\}(p_1'(\eta))^{\mu}(p_2'(\eta))^{M-\mu} \equiv 0 \quad (8)$$

but now with the additional property that $S$ is minimal with equation (8). Let $S$ have $L$ elements.

Since we may assume that the polynomials $p_i(\eta)$ are nonconstant, we know that the $p_i'(\eta)$ are not identically zero and hence that $L \geqslant 2$.

Let $\psi_1, \ldots, \psi_L$ be some enumeration of the summands that occur in equation (8). Then, after factoring out possible common zeroes of the entire holomorphic functions $\psi_1, \ldots, \psi_L$ we get an entire holomorphic curve

$$\Psi: \mathbb{C} \to \mathbb{P}^{L-1}; \eta \to [\psi_1(\eta) : \cdots : \psi_L(\eta)].$$

We claim that $\Psi$ is a polynomial map, i.e., all exponential factors of the $\psi_i$ are equal (up to constant factors, of course).[1]

Let first $L = 2$. If $\Psi$ is not polynomial, we would get, by dividing in equation (8) through one of the exponential terms, that the exponential of a nonconstant polynomial is equal to a quotient of two other polynomials, which is absurd.

Let now $L \geqslant 3$. If we denote the homogeneous coordinates of $\mathbb{P}^{L-1}$ by $[z_1 : \cdots : z_L]$, the image of $\Psi$ is contained in the hyperplane $H = \{z_1 + \cdots + z_L = 0\}$. So we can regard $\Psi$ also as a holomorphic curve contained in $H$ (which is isomorphic to $\mathbb{P}^{L-2}$). It is now important that the hyperplanes $H_i = \{z_i = 0\} \cap H$ are in general position in $H$, and that the entire curve $\Psi$ does not map $\mathbb{C}$ entirely into any hyperplane in $H$. (The latter follows from the minimality condition in equation (8).) Hence we can apply the Second Main Theorem (of Section 2), which

[1] We could immediately finish up the proof by using a Second Main Theorem for moving targets to equation (8), as to be found, e.g., in the paper of Ru and Stoll [25]. Another approach is to treat equation (8) directly with a generalized Borel's theorem. (We can regard this equation as a sum of nonvanishing holomorphic functions with coefficients which may vanish, but which grow of a smaller order than the nonvanishing functions, only.) We present here a more elementary argument based on the Second Main Theorem which might also be considered somewhat simpler.

yields

$$(L-(L-2)-1)T(\Psi, r) \leqslant \sum_{i=1}^{L} N_{\Psi}(H_i, r) + O(\log r), \tag{9}$$

because the entire curve $\Psi$ is of finite order at most two by Lemma 2.1. Now we have

$$N_{\Psi}(H_i, r) = N(\{\psi_i = 0\}, r) \leqslant M(N(\{p_1' = 0\}, r) + N(\{p_2' = 0\}, r)).$$

The First Main Theorem (of Section 2) and Lemma 2.3 imply that the right-hand side grows at most of order $O(\log r)$, so equation (9) yields that

$$T(\Psi, r) = O(\log r). \tag{10}$$

We apply Lemma 2.3 to $[\psi_i : \psi_j]: \mathbb{C} \to \mathbb{P}_1$: Since its characteristic function is bounded by $\log r$ according to (1) and (10), the quotient $\psi_i/\psi_j$ equals the quotient of two polynomials. Altogether $\Psi$ is a polynomial map. So in equation (8) we can factor out the exponential factors, and get a homogeneous polynomial equation.

We want to show now that there exist nonvanishing complex numbers $\gamma$ and $\lambda$ such that

$$\lambda p_1'(\eta) = \gamma p_2'(\eta). \tag{11}$$

We only need to know that $p_1'(\eta)$ and $p_2'(\eta)$ are linearly dependent, because if one of these is the zero polynomial, we have algebraic degeneracy of $g$ and hence of $f$. So assume that $p_1'(\eta)$ and $p_2'(\eta)$ are linearly independent. Then no linear combination of $p_1(\eta)$ and $p_2(\eta)$ is a constant polynomial. So we get that for all $(\mu, j, k) \in S$ the terms $\mu + j$ in the summands

$$c_{\mu jk} a_{\mu jk} \exp((\mu + j)p_1(\eta)) + (M - \mu + k)p_2(\eta))(p_1'(\eta))^{\mu}(p_2'(\eta))^{M-\mu}$$

are equal, and also the terms $k + (M - \mu)$ are the same as well. But then for a given $\mu_0$ there can be at most one $(\mu_0, j, k) \in S$. So we get a nontrivial homogenous equation of degree $M$ in $p_1'(\eta)$ and $p_2'(\eta)$, which then can be factored in linear factors. Since then one of the linear factors has to vanish identically, we get the linear dependency of $p_1'(\eta)$ and $p_2'(\eta)$ again, so the assumption of linear independency was wrong.

We now want to construct a special symmetric form with at most logarithmic poles as singularities along the curve $C$ which is annihilated by $f$.

Let us simply state equation (11) in terms of the original entire curve $f$. We have

$$p_i'(\eta) = \frac{dg_i(\eta)}{g_i(\eta)} = (\Phi \circ f)^* \frac{d\xi_i}{\xi_i} = f^* \omega_i \tag{12}$$

where $\omega_i$; $i = 1, 2$ is a differential one-form on $\overline{X}$ with at most logarithmic poles along $C$ Define $\omega_0 = \lambda\omega_1 - \gamma\omega_2$ Then $\omega_0 \in H^0(\overline{X}, E^*)$, and since

$$\omega_0 = \Phi^*\left(\lambda \frac{d\xi_1}{\xi_1} - \gamma \frac{d\xi_2}{\xi_2}\right)$$

and the map $\Phi$ is a local isomorphism outside the branching, we have

$$\omega_0 \neq 0.$$

Furthermore, by equations (12) and (11) we have

$$f^*\omega_0 \equiv 0.$$

Now the proof of the fact that $f$ is algebraically degenerate is almost finished: Let $\sigma \in H^0(\mathbb{P}(E), \mathcal{O}(\mu) \otimes p^*[-k_0 D])$ be the section constructed in Lemma 3.6, and $\tilde{\sigma} \in H^0(\mathbb{P}(E), \mathcal{O}(1))$ the section which corresponds to $\omega_0$ We recall that both sections are nontrivial, that $\mu \geqslant 2$, and that $V(\sigma)$ contains only one horizontal component, which we will denote by $S_\sigma$, with multiplicity one. We also recall that, by Theorem 3 1, the lift of $f$ to $\mathbb{P}(E)$, which we denoted by $F$, maps entirely into $V(\sigma)$. We may assume that it maps into $S_\sigma$; otherwise, by projecting down to $\overline{X}$ we get that $f$ is algebraically degenerate and we are done.

If $\tilde{\sigma}$ does not vanish identically on $S_\sigma$, $F$ maps into the zero set of $\tilde{\sigma}$ in $S_\sigma$, which has codimension at least two. So projecting down to $\overline{X}$ again yields algebraic degeneracy of $f$

Hence we now may assume that $\tilde{\sigma}$ vanishes identically on $S_\sigma$ Since $\tilde{\sigma} \in H^0(\mathbb{P}(E), \mathcal{O}(1))$, the degree of $V(\tilde{\sigma})$ with respect to a generic fiber of the map $p$: $\mathbb{P}(E) \to \overline{X}$ is one (cf. the argument in the proof of Lemma 3.5). However, since $S_\sigma$ is the only horizontal component of the zero set of $\sigma \in H^0(\mathbb{P}(E), \mathcal{O}(\mu) \otimes p^*[-k_0 D])$ with $\mu \geqslant 2$ and has multiplicity one, and since $\tilde{\sigma}$ vanishes on $S_\sigma$, the degree of $V(\tilde{\sigma})$ with respect to such a generic fiber must be at least two, which is a contradiction So this case cannot occur and the proof of Theorem 4 2 is complete. □

*Proof of Lemma 4 4.* The assertion $g^*\Omega \equiv 0$ is clear from $f^*\omega \equiv 0$ and the construction of $\Omega$

In order to prove the assertion $\Omega \not\equiv 0$, we choose a point $\xi^0 \in \mathbb{P}_2 \backslash (B \cup \{w_0 = 0\})$. In a small neighborhood $U(\xi^0)$ we have the $N$ biholomorphic functions $a_i(\xi)$, $i = 1, \ldots, N$ which invert the map $\Phi$ on $U(\xi^0)$. Then we have

$$((a_i)^*(\omega))(\xi) = \sum_{j=0}^{m} b_{ij}(\xi)(d\xi_1)^j (d\xi_2)^{m-j}. \tag{13}$$

After possibly moving the point $\xi^0$ in $U(\xi^0)$, we may assume that the meromorphic functions $b_{ij}(\xi)$ either vanish identically on $U(\xi^0)$ or have no zero or singularity in $\xi^0$. Let now for each $i = 1, \ldots, N$ the index $j(i)$ be the maximal $j \in \{0, \ldots, m\}$ such that $b_{ij}(\xi^0) \neq 0$. Let $k = \sum_{i=1}^N j(i)$. Then the $(d\xi_1)^k(d\xi_2)^{M-k}$-monomial of $\Omega$ in the point $\xi_0$ is equal to $\prod_{i=1}^N b_{ij(i)}(\xi^0)$, which is not equal to zero by construction

Last we have to show that $\Omega$ extends to a rational symmetric $M$-form on $\mathbb{P}_2$. We only have to show how $\Omega$ can be extended over smooth points of the branching locus, because by Levi's extension theorem (cf. [13]) we then can extend it over the singular locus which is of codimension two. Then, by Chow's theorem it is rational. So assume $P \in B$ is a smooth point of $B$. Then (cf. [12]) there exists a neighborhood of $P$ over which $\Phi$ is an analytically branched covering of a very special form: for every point $Q$ over $P$ one can introduce local coordinates $\xi_1, \xi_2$ around $P$ and $z_1, z_2$ around $Q$ such that $\xi_1(P) = \xi_2(P) = z_1(Q) = z_2(Q) = 0$, and neighborhoods $U = \{|\xi_1| < 1, |\xi_2| < 1\}$, $V = \{|z_1| < 1, |z_2| < 1\}$ such that, for some $b \in \{1, \ldots, N\}$, we have

$$\Phi: V \to U; (z_1, z_2) \to (z_1^b, z_2) \tag{14}$$

In order to prove our assertion in a neighborhood of $P \in B$, it is sufficient to prove it for the analytically branched covering in equation (14).

For $k = 0, \ldots, b-1$, let

$$g_k: V \to V; (z_1, z_2) \to \left(\exp\left(\frac{2\pi ik}{b}\right) z_1, z_2\right)$$

Then $G = \{g_0, \ldots, g_{b-1}\}$ is just the group of deck transformations, i.e., automorphisms which respect $\Phi$. For the meromorphic symmetric $m$-form $\omega$ on $V$, let $\tilde{\Omega}$ be the symmetric product of the $b$ meromorphic symmetric $m$-forms $(g_i)^*(\omega)$ on $V$. We are done, if we show that by projecting down with $\Phi$ this form gives rise to a meromorphic symmetric $M$-form on $U$, because in $U \backslash B$ this is just the form $\Omega$. The symmetric $M$-form $\tilde{\Omega}$ can be uniquely written in the form

$$\tilde{\Omega}(z_1, z_2) = \sum_{i=0}^{M} r_{ij}(z_1, z_2) \left(\frac{dz_1}{z_1}\right)^i (dz_2)^{M-i} \tag{15}$$

with meromorphic functions $r_{ij}$ in the variables $(z_1, z_2)$. Now $\tilde{\Omega}(z_1, z_2)$ is invariant under the action of $G$, $(dz_1/z_1)^i$ and $(dz_2)^{M-i}$ are also $G$-invariant. Moreover, $b(dz_1/z_1) = d\xi_1/\xi_1$ and $dz_2 = d\xi_2$. Hence the $r_{ij}$ are $G$-invariant functions, i.e., these are pullbacks of meromorphic functions on $U$. □

**5. Application to the projective plane and complete intersections.** We shall apply Theorem 4.2 Throughout this section, we make the following assumptions:

Let the smooth complex surface $\overline{X}$ be a complete intersection

$$\overline{X} = V_2^{(a_1 \ldots a_r)} \subset \mathbb{P}_{r+2}, r \geqslant 0$$

of hypersurfaces of degrees $a_j$, $j = 1, \ldots, r$ in $\mathbb{P}_{r+2}$. Set $A = \prod_{i=1}^{r} a_i$ and $a = \sum_{i=1}^{r} a_i$. We note that the case $\overline{X} = \mathbb{P}_2$ is given by setting $r = 0$, $a = 0$, $A = 1$. Let smooth curves $C_j$, $j = 1, 2, 3$ be given in $\overline{X}$ which intersect in normal crossings, and let $C = C_1 \cup C_2 \cup C_3$. We assume that the curves $C_j$ are transversal intersections of $\overline{X}$ with hypersurfaces of degrees $b_j$. We set $b = b_1 + b_2 + b_3$. Let again $X = \overline{X} \backslash C$.

LEMMA 5.1. *The Euler numbers of $\overline{X}$ and $C_j$ are*

$$e(\overline{X}) = A(2 + (a - r - 1)^2)$$

*and*

$$e(C_j) = Ab_j(3 + r - a - b_j)$$

*Proof.* This is a direct consequence of the Hirzebruch-Riemann-Roch theorem. According to [17], Theorem 22.1.1, the $\chi_y$-characteristic of a complete intersection can be computed from a generating function. Its value at $y = -1$ yields the Euler number. □

In order to determine when the assumptions of Theorem 4.2 are satisfied, we first compute $\bar{c}_1^2(X) - \bar{c}_2(X)$.

PROPOSITION 5.2. *In the above situation,*

$$\bar{c}_1^2(X) - \bar{c}_2(X) = A\left((a - r - 3)(b - 4) - 6 + \sum_{i<j} b_i \cdot b_j\right),$$

*and*

$$\det(E^*) = (a + b - 3 - r)\tilde{H},$$

*where $\tilde{H}$ is a hyperplane section.*

*Proof.* According to a result of Sakai [26], we have $\det(E^*) = [\Gamma]$, where $\Gamma = K_{\overline{X}} + C$. Then the second claim follows from the adjunction formula. Furthermore (cf. [26]),

$$c_1^2(E) - c_2(E) = c_1^2(E^*) - c_2(E^*) = \Gamma^2 - e(\overline{X} \backslash C) = \Gamma^2 - e(\overline{X}) + e(C),$$

where $\Gamma^2$ denotes the self-intersection. It equals

$$\Gamma^2 = A(a + b - r - 3)^2.$$

For the Euler number of $\overline{X}$, we use Proposition 5.1. The Euler number of $C$ is evaluated in terms of the Euler numbers $e(C_j)$ of the components and the respective intersection numbers

$$C_i \cdot C_j = Ab_ib_j$$

to be

$$e(C) = \sum_{j=1}^{3} e(C_j) - \sum_{i<j} C_i \cdot C_j.$$

From these equalities we get immediately the above formula for $c_1^2 - c_2$. □

Now Theorem 4.2 yields the following.

THEOREM 5.3. *Any entire holomorphic curve $f: \mathbb{C} \to X$ of order at most two is algebraically degenerate, if*

(i) $\operatorname{Pic}(\overline{X}) = \mathbb{Z}$

(ii) $(a - r - 3)(b - 4) + \sum_{i<j} b_ib_j > 6$.

The proof follows from Theorem 4.2 and Proposition 5.2. Observe that $a + b \geqslant r + 3$ is automatic.

THEOREM 5.4. *Any entire holomorphic curve of order at most two $f: \mathbb{C} \to X$ is algebraically degenerate in any of the following cases:*

(a) $\operatorname{Pic}(\overline{X}) = \mathbb{Z}$, *and* $a \geqslant r + 3$, $b \geqslant 5$.

(b) $\overline{X} \subset \mathbb{P}_3$ *is a "generic" hypersurface of degree at least four, and* $b \geqslant 5$.

(c) *Let* $\overline{X} = \mathbb{P}_2$. *Let* $b_1, b_2, b_3 \geqslant 2$ *and at least one* $b_j \geqslant 3$, *or up to enumeration* $b_1 = 1, b_2 \geqslant 3, b_3 \geqslant 4$.

*Remark.* "Generic" indicates the complement of a countable union of proper varieties in space of all hypersurfaces.

*Proof.* Case (a) is obvious. Case (b) is an application of the Noether-Lefschetz theorem [6] and case (a). For case (c) we set, e.g., $r = a = 0$, $A = 1$. Then

$$\bar{c}_1^2(X) - \bar{c}_2(X) = -3b + 6 + \sum_{i<j} b_i \cdot b_j$$

is equal to

$$(b_1 - 2)(b_2 - 2) + (b_1 - 2)(b_3 - 2) + (b_2 - 2)(b_3 - 2) + b - 6$$

or to

$$(b_1 - 1)(b_2 - 1) + (b_1 - 1)(b_3 - 2) + (b_2 - 3)(b_3 - 4) + (2b_2 + b_3) - 9.$$

From these facts the assertion of case (c) follows immediately. □

## 6. Algebraic degeneracy of entire curves versus hyperbolicity

THEOREM 6.1. *Let $C$ be the union of three smooth curves $C_j$ $j = 1, 2, 3$ in $\mathbb{P}_2$ of degree $d_j$ with*

$$d_1, d_2, d_3 \geqslant 2 \text{ and at least one } d_j \geqslant 3.$$

*Then for generic such configurations, $\mathbb{P}^2 \backslash C$ is complete hyperbolic and hyperbolically embedded in $\mathbb{P}_2$.*

*More precisely, the conclusion holds when the curves intersect only in normal crossings and, for the case where one curve is a quadric, there further does not exist a line which intersects each of the other two curves in only one point that also lies on the quadric.*

*Proof.* In order to prove that $\mathbb{P}_2 \backslash C$ is hyperbolic and hyperbolically embedded in $\mathbb{P}_2$, we only will have to prove, by an easy corollary of a theorem of M. Green (cf. [15]), that there does not exist a nonconstant entire curve $f: \mathbb{C} \to \mathbb{P}_2 \backslash C$ of order at most two.

We know from Theorem 5.4 that the entire curve $f: \mathbb{C} \to \mathbb{P}_2 \backslash C$ of order at most two is contained in an algebraic curve $A \subset \mathbb{P}_2$ of degree $d_0$, say.

Now the proof is almost the same as in [8]. Assume that there exists an irreducible algebraic curve $A \subset \mathbb{P}_2$ such that $A \backslash C$ is not hyperbolic. We know that $A \cap C$ consists of at least two points $P$ and $Q$. Moreover, $A$ cannot have a singularity at $P$ or $Q$ with different tangents, because $A$ has to be reducible in such a point, and $A \backslash C$ could be identified with an irreducible curve with at least three punctures. (This follows from blowing up such a point or considering the normalization.)

So $A \cap C$ consists of exactly two points $P$ and $Q$ with simple tangents. We denote the multiplicities of $A$ in $P$ and $Q$ by $m_P$ and $m_Q$. Then the inequality (cf. [10])

$$m_P(m_P - 1) + m_Q(m_Q - 1) \leqslant (d_0 - 1)(d_0 - 2)$$

implies

$$m_P, m_Q < d_0 \text{ or } d_0 = m_P = m_Q = 1 \tag{16}$$

After a suitable enumeration of its components, we may assume that $P \in C_1 \cap C_2$ and $Q \in C_3$. If $Q \notin C_2 \cup C_1$, we are done, since then we may assume that $A$ is not tangential to $C_2$, and then, computing intersection multiplicities according to [10], we have

$$m_P = I(P, A \cap C_2) = d_2 d_0,$$

which contradicts equation (16). So we may assume that $Q \in C_2 \cap C_3$. Now $A$ has to be tangential to $C_1$ in $P$ and to $C_3$ in $Q$; otherwise we again get $m_P = d_1 d_0$ or

$m_Q = d_3 d_0$, which contradicts equation (16). But then $C_2$ is not tangential to $A$ in $P$ or $Q$, so we have

$$m_P + m_Q = I(P, A \cap C_2) + I(Q, A \cap C_2) = d_2 d_0.$$

Again by equation (16) this is only possible if $d_2 = 2$ and $m_P = m_Q = d_0 = 1$, but then we are in a situation which we excluded in Theorem 6.1, which is a contradiction. □

We make the same assumptions as in Section 5.

THEOREM 6.2. *Let $\overline{X} \subset \mathbb{P}_3$ be a "generic" smooth hypersurface of degree $d \geqslant 5$ and $b \geqslant 5$. Then $X = \overline{X} \backslash C$ is hyperbolic and hyperbolically embedded in $\overline{X}$.*

*Proof.* According to Xu [29] and Clemens [7], $\overline{X}$ does not contain any rational or elliptic curves. Hence Theorem 5.4 yields the claim. □

## REFERENCES


[1] Y. ADACHI AND M. SUZUKI, *On the family of holomorphic mappings into projective space with lacunary hypersurfaces*, J. Math. Kyoto Univ. **30** (1990), 451–458.

[2] ———, "Degeneracy points of the Kobayashi pseudodistances on complex manifolds" in *Several Complex Variables and Complex Geometry, Part 2*, Proc. Sympos. Pure Math. **52**, Amer. Math. Soc., Providence, 1991, 41–51.

[3] V. A. BABETS, *Theorems of Picard type for holomorphic mappings*, Siberian Math. J. **25** (1984), 195–200.

[4] W. BARTH, C. PETERS, AND A. VAN DE VEN, *Compact Complex Surfaces*, Ergeb. Math. Grenzgeb. (3) **4**, Springer-Verlag, Berlin, 1984.

[5] F. A. BOGOMOLOV, *Holomorphic tensors and vector bundles on projective varieties*, Math. USSR-Izv. **13** (1979), 499–555.

[6] J. CARLSON, M. GREEN, P. GRIFFITHS, AND J. HARRIS, *Infinitesimal variations of Hodge structure (I)*, Compositio Math. **50** (1983), 109–205.

[7] H. CLEMENS, J. KOLLÁR, AND S. MORI, *Higher-dimensional complex geometry*, Asterisque **166** (1988).

[8] G. DETHLOFF, G. SCHUMACHER, AND P. M. WONG, *Hyperbolicity of the complements of plane algebraic curves*, to appear in Amer. J. Math.

[9] A. EREMENKO AND M. SODIN, *The value distribution of meromorphic functions and meromorphic curves from the point of view of potential theory*, St. Petersburg Math. J. **3** (1992), 109–136.

[10] W. FULTON, *Algebraic Curves*, Benjamin, New York, 1969.

[11] H. GRAUERT, *Jetmetriken und hyperbolische Geometrie*, Math. Z. **200** (1989), 149–168.

[12] H. GRAUERT AND R. REMMERT, *Komplexe Räume*, Math. Ann. **136** (1958), 245–318.

[13] ———, *Coherent Analytic Sheaves*, Grundlehren Math. Wiss. **265**, Springer-Verlag, Berlin, 1984.

[14] M. GREEN, *Some Picard theorems for holomorphic maps to algebraic varieties*, Amer. J. Math. **97** (1975), 43–75.

[15] ———, *The hyperbolicity of the complement of $2n + 1$ hyperplanes in general position in $\mathbb{P}_n$ and related results*, Proc. Amer. Math. Soc **66** (1977), 109–113.

[16] W. K. HAYMAN, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[17] F. HIRZEBRUCH, *Topological Methods in Algebraic Geometry*, 3rd ed., Grundlehren Math. Wiss. **131**, Springer-Verlag, New York, 1966.

[18] S. KOBAYASHI, *Hyperbolic manifolds and holomorphic mappings*, Marcel Dekker, New York, 1970.

[19] S. KOBAYASHI AND T. OCHIAI, *On complex manifolds with positive tangent bundles*, J. Math. Soc. Japan **22** (1970), 499–525.

[20] S. LANG, *Hyperbolic and diophantine analysis*, Bull. Amer. Math. Soc. (N.S.) **14** (1986), 159–205

[21] ———, *Introduction to Complex Hyperbolic Spaces*, Springer-Verlag, New York, 1987

[22] S. LU, "On meromorphic maps into varieties of log-general type" in *Several Complex Variables and Complex Geometry Part 2*, Proc. Sympos. Pure Math. **52**, Amer. Math. Soc., Providence, 1991, 305–333.

[23] S. LU AND S. T. YAU, *Holomorphic curves in surfaces of general type*, Proc. Nat. Acad. Sci. USA **87** (1990), 80–82

[24] Y. MIYAOKA, *On the Chern numbers of surfaces of general type*, Invent. Math. **42** (1977), 225–237

[25] M. RU AND W. STOLL, *The second main theorem for moving targets*, J. Geom. Anal. **1** (1991), 99–138.

[26] F. SAKAI, *Semi-stable curves on algebraic surfaces and logarithmic pluricanonical maps*, Math. Ann. **254** (1980), 98–120.

[27] B. SHIFFMAN, *On holomorphic curves and meromorphic maps in projective space*, Indiana Univ. Math. J. **28** (1979), 627–641.

[28] P. M. WONG, *On the second main theorem of Nevanlinna theory*, Amer. J. Math. **111** (1989), 549–583.

[29] G. XU, *Subvarieties of general hypersurfaces in projective space*, preprint, 1992.

[30] M. ZAIDENBERG, *Stability of hyperbolic embeddedness and construction of examples*, Math. USSR-Sb. **63**: 2 (1989), 351–361

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