UNIQUENESS PROBLEM FOR MEROMORPHIC MAPPINGS WITH TRUNCATED MULTIPLICITIES AND MOVING TARGETS

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Abstract

In this paper, using techniques of value distribution theory, we give a uniqueness theorem for meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ with $(3n+1)$ moving targets and truncated multiplicities.

1 Introduction

The uniqueness problem of meromorphic mappings under a condition on the inverse images of divisors was first studied by R. Nevalinna [6]. He showed that for two nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$, if they have the same inverse images for five distinct values, then $f \equiv g$, and that g is a special type of linear fractional tranformation of f if they have the same inverse images, counted with multiplicities, for four distinct values. These results were generalized to the case of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ by H. Fujimoto [1].

In the last years, this problem was continued to be studied by H.Fujimoto[2], [3], L. Smiley [10], S. Ji [5], M. Ru [9], Z. Tu [11].

Let f, a be two meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ with reduced representations $f = (f_0 : \cdots : f_n)$, $a = (a_0 : \cdots : a_n)$. Set $(f, a) := a_0f_0 + \cdots + a_nf_n$. We say that a is "small" with respect to f if $T_a(r) = \circ(T_f(r))$ as $r \to \infty$ (outside a set of finite Lebesgue measure). Assume that $(f, a) \not\equiv 0$, denote by $v_{(f,a)}$ the map of $\mathbb{C}^m$ into $\mathbb{N}_0$ with $v_{(f,a)}(z) = 0$ if $(f,a)(z) \neq 0$ and $v_{(f,a)}(z) = k$ if z is a zero point of (f,a) with multiplicity k.

Let $a_1, \dots, a_q \quad (q \geq n+1)$ be meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ with reduced representations $a_j = (a_{j0} : \cdots : a_{jn}), \quad j = 1, \dots, q$. We say that $\{a_j\}_{j=1}^q$ are in general position if for any $1 \leq j_0 < \cdots < j_n \leq q$, $\det(a_{j_k i}, 0 \leq k, i \leq n) \not\equiv 0$.

For each $j \in \{1, ..., q\}$, we put $\widetilde{a_j} = (\frac{a_{j0}}{a_{jt_j}} : ... : \frac{a_{jn}}{a_{jt_j}})$ and $(f, \widetilde{a_j}) = f_0 \frac{a_{j0}}{a_{jt_j}} + ... + f_n \frac{a_{jn}}{a_{jt_j}}$, where a_{jt_j} is the first element of $a_{j0}, ..., a_{jn}$ not identically equal to zero.

Let $\mathcal{M}$ be the field (over $\mathbb{C}$) of all meromorphic functions on $\mathbb{C}^m$. Denote by $\mathcal{R}\Big(\{a_j\}_{j=1}^q\Big) \subset \mathcal{M}$ the subfield generated by the set $\{\frac{a_{ji}}{a_{jt_j}}, 0 \leq i \leq n, 1 \leq j \leq q\}$ over $\mathbb{C}$. Define $\widetilde{\mathcal{R}}\Big(\{a_j\}_{j=1}^q\Big) \subset \mathcal{M}$ to be the subfield over $\mathbb{C}$ which is generated by all $h \in \mathcal{M}$ with $h^k \in \mathcal{R}\Big(\{a_j\}_{j=1}^q\Big)$ for some integer k. These subfields are independant of the reduced representations $a_j = (a_{j0} : \cdots : a_{jn})$, $j = 1, \dots, q$, and they are of course also independant of our choice of the a_{jt_j}, because they contain all quotients of the quotients $\frac{a_{ji}}{a_{jt_j}}, i = 0, \dots, n$.

We say that f is linearly nondegenerate over $\mathcal{R}\Big(\{a_j\}_{j=1}^q\Big)$ (respectively $\widetilde{\mathcal{R}}\Big(\{a_j\}_{j=1}^q\Big)$) if $f_0, \dots, f_n$ are linearly independent over $\mathcal{R}\Big(\{a_j\}_{j=1}^q\Big)$ (respectively $\widetilde{\mathcal{R}}\Big(\{a_j\}_{j=1}^q\Big)$).

Let $f, g : \mathbb{C}^m \longrightarrow \mathbb{C}P^n$ be two nonconstant meromorphic mappings and $\{a_j\}_{j=1}^q$ be q "small" (with respect to f) meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ in general position such that $(f, a_j) \not\equiv 0$, $(g, a_j) \not\equiv 0$, $j = 1, \dots, q$. Set $\widetilde{E}_f^j := \{z \in \mathbb{C}^m : v_{(f,a_j)}(z) > 0\}$. Assume that:

i) $v_{(f,a_j)} = v_{(g,a_j)}$ for all $j \in \{1, \dots, q\}$

ii) $\dim\big(\widetilde{E}_f^i \cap \widetilde{E}_f^j\big) \leq m-2$ for all $1 \leq i < j \leq q$, and

iii) $f = g$ on $\bigcup_{j=1}^q \widetilde{E}_f^j$.

In [11] Z. Tu showed that:

Theorem A. If $q = 3n+1$ and f is linearly nondegenerate over $\mathcal{R}\Big(\{a_j\}_{j=1}^q\Big)$, then there exists a $(n+1)\times(n+1)$-matrix L with elements in $\widetilde{\mathcal{R}}\Big(\big\{a_j\big\}_{j=1}^q\Big)$ and $\det(L)\not\equiv 0$ such that $L\cdot f = g$.

Theorem B. If $q = 3n+2$ and f is linearly nondegenerate over $\widetilde{\mathcal{R}}\Big(\big\{a_j\big\}_{j=1}^q\Big)$ then $f = g$.

These theorems (without conditions ii) and iii)) were first showed by H. Fujimoto ([1]) for hyperplanes ($\big\{a_j\big\}_{j=1}^q$ are constant).

In the above Theorems multiplicities are not truncated (we say that multiplicities are truncated by a positive integer M if i) is replaced by the following: $\min\big\{v_{(f,a_j)}, M\big\} = \min\big\{v_{(g,a_j)}, M\big\}$). However, the uniqueness problem with truncated multiplicities was studied in [2], [3], [5], [10] for hyperplanes ($\big\{a_j\big\}_{j=1}^q$ are constant) and in [9] for moving targets.

For hyperplanes, in [10] L. Smiley proved Theorem B with multiplicities are truncated by 1, and in [2], [3] H. Fujimoto gave some results related to Theorem B with multiplicities are truncated by a positive integer M.

For moving targets, in [9] M. Ru gave some results related to Theorem B with multiplicities are truncated by 1, but where the number $q = 3n+2$ is replaced by bigger one.

The main purpose of this paper is to give uniqueness theorems for the case of $3n+1$ moving targets and multiplicities which are truncated by a positive integer M. Our results are improvements of Theorems A-B where the number $q = 3n+2$ is replaced by smaller one, the multiplicities are truncated and the condition iii) is replaced by weaker one. In particular, we prove that for $n \geq 2$ we get $f = g$ already for $q = 3n+1$.

The proofs of our results are applications of a generalized Borel Lemma: For the case where multiplicities are truncated, our object does not satisfy the assumption “nowhere vanishing holomorphic functions” of the (classical) Borel Lemma. So, first of all, using the techniques of value distribution theory, we give Lemma 3.1, which is a generalization of the Borel Lemma for meromorphic functions.

In order to show that under the conditions of our uniqueness theorems the assumption of Lemma 3.1 is satisfied, we need some results of value distribution theory of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ for moving targets. But the Second Main Theorem as in [8] (where multiplicities are not truncated) or as in [11] (where multiplicities are truncated by a positive integer ℓ) seems to be not sufficient for our purpose. In order to overcome this difficulty we establish a Second Main Theorem for meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ for $(n+2)$ moving targets with multiplicities truncated by n.

Our main results are as follows:

Let $f, g : \mathbb{C}^m \longrightarrow \mathbb{C}P^n$ be two nonconstant meromorphic mappings and $\big\{a_j\big\}_{j=1}^{3n+1}$ be "small" (with respect to f) meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ in general position such that $(f, a_j) \not\equiv 0$, $(g, a_j) \not\equiv 0$, $j = 1, \ldots, 3n+1$. Put $M = 6n(n+1)[N^2(N-1)+1]$, where $N = \begin{pmatrix} 2n+2 \\ n+1 \end{pmatrix}$.

Set $E_f^j := \big\{z \in \mathbb{C}^m : 0 \leq v_{(f,a_j)}(z) \leq M\big\}$, ${}^*E_f^j := \big\{z \in \mathbb{C}^m : 0 < v_{(f,a_j)}(z) \leq M\big\}$ and similarly for E_g^j, ${}^*E_g^j$, $j = 1, \ldots, 3n+1$.

Assume that:

i) $v_{(f,a_j)} = v_{(g,a_j)}$ on $E_f^j \cap E_g^j$ for all $j \in \{1, \ldots, 3n+1\}$.

ii) $\dim\big({}^*E^i \cap {}^*E^j\big) \leq m-2$ for all ${}^*E^i \in \big\{{}^*E_f^i, {}^*E_g^i\big\}$, ${}^*E^j \in \big\{{}^*E_f^j, {}^*E_g^j\big\}$ and for all $i \neq j$ with $i \in \{1, \ldots, n+3\}, j \in \{1, \ldots, 3n+1\}$.

iii) $f = g$ on $\bigcup\limits_{i=1}^{n+4} \big({}^*E_f^i \cap {}^*E_g^i\big)$ for $n \geq 2$.

This means in particular that in i), ii) and iii) we do not need to pay attention to points where $v_{(f,a_j)}$ or $v_{(g,a_j)}$ is bigger than M.

Theorem 1.

1) If $n = 1$ *and* f, g *are linearly nondegenerate over* $\mathcal{R}\Big(\big\{a_j\big\}_{j=1}^{4}\Big)$ *then there exists a* 2×2 *- matrix* L *with elements in* $\widetilde{\mathcal{R}}\Big(\big\{a_j\big\}_{j=1}^{4}\Big)$ *and* $\det(L) \not\equiv 0$ *such that* $L \cdot f = g$.

2) If $n \geq 2$ and f, g *are linearly nondegenerate over* $\mathcal{R}\Big(\big\{a_j\big\}_{j=1}^{3n+1}\Big)$ *then* $f = g$.

We remark that in the case $n = 1$, we cannot omit the matrix L, as can be seen easily as follows: Let $f : \mathbb{C} \to \mathbb{C}$ a nonconstant nonvanishing holomorphic function, then consider the two functions f, $1/f$ and the four values $0, \infty, 1, -1$. Note also that condition i) is weaker than a truncated multiplicities condition.

We give the following theorem for the case where multiplicities are truncated.

Theorem 2. Let $f, g : \mathbb{C}^m \longrightarrow \mathbb{C}P^n$ be two nonconstant meromorphic

mappings and $\{a_j\}_{j=1}^{3n+1}$ be "small" (with respect to f) meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ in general position such that $(f, a_j) \not\equiv 0$, $(g, a_j) \not\equiv 0$, $j = 1, \ldots, 3n+1$. Put $M = 3n(n+1)N^2(N-1)+(3n+4)n$, where $N = \binom{2n+2}{n+1}$.
Set ${}^*E_f^j := \{z \in \mathbb{C}^m : 0 < v_{(f,a_j)}(z) \leq M\}$, $j = 1, \ldots, 3n+1$.
Assume that:

i) $\min\{v_{(f,a_j)}, M\} = \min\{v_{(g,a_j)}, M\}$ for all $j \in \{1, ..., 3n+1\}$.

ii) $\dim({}^*E_f^i \cap {}^* E_f^j) \leq m-2$ for all $i \neq j$ with $i \in \{1, ..., n+3\}, j \in \{1, ..., 3n+1\}$.

iii) $f = g$ on $\bigcup_{i=1}^{n+4} {}^*E_f^i$ for $n \geq 2$.

Then :

1) If $n = 1$ and f is linearly nondegenerate over $\mathcal{R}(\{a_j\}_{j=1}^4)$ then there exists a 2×2 - matrix L with elements in $\widetilde{\mathcal{R}}(\{a_j\}_{j=1}^4)$ and $\det(L) \not\equiv 0$ *such that $L \cdot f = g$.*

2) If $n \geq 2$ and f is linearly nondegenerate over $\mathcal{R}(\{a_j\}_{j=1}^{3n+1})$ then $f = g$.

We finally remark that in order to obtain uniqueness theorems with *fixed* targets only, the authors showed in [13] that the number $q = 3n+1$ of targets can be decreased and that one can use much smaller truncations.

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2 Preliminaries

We set $\|z\| = \big(|z_1|^2 + \cdots + |z_m|^2\big)^{1/2}$ for $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ and define

$$B(r) := \{z \in \mathbb{C}^m : |z| < r\}, \quad S(r) := \{z \in \mathbb{C}^m : |z| = r\} \text{ for all } 0 < r < \infty.$$

Define $d^c := \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial)$, $\upsilon := (dd^c\|z\|^2)^{m-1}$ and

$$\sigma := d^c\log\|z\|^2 \wedge (dd^c\log\|z\|^2)^{m-1}.$$

Let F be a nonzero holomorphic function on $\mathbb{C}^m$. For each $a \in \mathbb{C}^m$, expanding F as $F = \sum P_i(z-a)$ with homogeneous polynomials P_i of degree i around a, we define

$$v_F(a) := \min\{i : P_i \not\equiv 0\}.$$

Let φ be a nonzero meromorphic function on $\mathbb{C}^m$. We define the map v_φ as follows: For each $z \in \mathbb{C}^m$, we choose nonzero holomorphic functions F and G on a neighborhood U of z such that $\varphi = \dfrac{F}{G}$ on U and $\dim\big(F^{-1}(0) \cap G^{-1}(0)\big) \leq m-2$, and then we put $v_\varphi(z) := v_F(z)$. Set $|v_\varphi| := \overline{\big\{z \in \mathbb{C}^m : v_\varphi(z) \neq 0\big\}}$.

Let k, M be positive integers or $+\infty$. Set

$${}^{\leq M}v_\varphi^{[k]}(z) = 0 \text{ if } \quad v_\varphi(z) > M \text{ and } {}^{\leq M}v_\varphi^{[k]}(z) = \min\{v_\varphi(z), k\} \text{ if } \quad v_\varphi(z) \leq M$$

$${}^{>M}v_\varphi^{[k]}(z) = 0 \text{ if } \quad v_\varphi(z) \leq M \text{ and } {}^{>M}v_\varphi^{[k]}(z) = \min\{v_\varphi(z), k\} \text{ if } v_\varphi(z) > M.$$

We define

$${}^{\leq M}N_\varphi^{[k]}(r) := \int_1^r \frac{{}^{\leq M}n(t)}{t^{2m-1}}dt$$

and

$${}^{>M}N_\varphi^{[k]}(r) := \int_1^r \frac{{}^{>M}n(t)}{t^{2m-1}}dt \qquad (1 \leq r < +\infty)$$

where

$${}^{\leq M}n(t) := \int_{|v_\varphi| \cap B(r)} {}^{\leq M}v_\varphi^{[k]}.\upsilon \quad \text{for} \quad m \geq 2 \text{ , } {}^{\leq M}n(t) := \sum_{|z| \leq t} {}^{\leq M}v_\varphi^{[k]}(z) \quad \text{for} \quad m = 1$$

$${}^{>M}n(t) := \int_{|v_\varphi| \cap B(r)} {}^{>M}v_\varphi^{[k]}.\upsilon \quad \text{for} \quad m \geq 2 \text{ , } {}^{>M}n(t) := \sum_{|z| \leq t} {}^{>M}v_\varphi^{[k]}(z) \quad \text{for} \quad m = 1.$$

Set $N_\varphi(r) := {}^{\leq\infty}N_\varphi^{[\infty]}(r)$, $N_\varphi^{[k]}(r) := {}^{\leq\infty}N_\varphi^{[k]}(r)$.

We have the following Jensen's formula (see [3], P.177):

$$N_\varphi(r) - N_{\frac{1}{\varphi}}(r) = \int_{S(r)} \log|\varphi|\sigma - \int_{S(1)} \log|\varphi|\sigma.$$

Let $f : \mathbb{C}^m \longrightarrow \mathbb{C}P^n$ be a meromorphic mapping. For arbitrary fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ of $\mathbb{C}P^n$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, which means that each f_i is a holomorphic function on $\mathbb{C}^n$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ outside the analytic set $\{f_0 = \cdots = f_n = 0\}$ of codimension ≥ 2. Set $\|f\| = \left(|f_0|^2 + \cdots + |f_n|^2\right)^{1/2}$.

The characteristic function of f is defined by

$$T_f(r) = \int\limits_{S(r)} \log\|f\|\sigma - \int\limits_{S(1)} \log\|f\|\sigma, \quad 1 \leq r < +\infty.$$

For a meromorphic function φ on $\mathbb{C}^m$, the proximity function is defined by

$$m(r, \varphi) := \int\limits_{S(r)} \log^+ |\varphi|\sigma$$

and we have, by the classical First Main Theorem that (see [4], p.135)

$$m(r, \varphi) \leq T_\varphi(r) + O(1).$$

Here, the characteristic function $T_\varphi(r)$ of φ is defined by considering φ as a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^1$.

We state the First and Second Main Theorem of Value Distribution Theory (see e.g.[11], [2]):

First Main Theorem. (Moving target version) *Let a be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ such that $(f, a) \not\equiv 0$. Then for reduced representations $f = (f_0 : \cdots : f_n)$ and $a = (a_0 : \cdots : a_n)$, we have:*

$$N_{(f,a)}(r) \leq T_f(r) + T_a(r) \quad \textit{for all} \quad r \geq 1.$$

For a hyperplane $H : a_0 w_0 + \cdots + a_n w_n = 0$ in $\mathbb{C}P^n$ with $\operatorname{im} f \not\subseteq H$, we denote $(f, H) = a_0 f_0 + \cdots + a_n f_n$, where $(f_0 : \cdots : f_n)$ again is a reduced representation of f.

Second Main Theorem. (Classical version) *Let f be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $H_1, \ldots, H_q$ $(q \geq n + 1)$ hyperplanes of $\mathbb{C}P^n$ in general position, then*

$$(q - n - 1)T_f(r) \leq \sum_{j=1}^{q} N^{[n]}_{(f,H_j)}(r) + o(T_f(r))$$

for all r *except for a set of finite Lebesgue measure.*

3 Proof of our results

First of all, we give a generalization of the Borel Lemma for meromorphic functions.

Lemma 3.1. Let $h_0, \dots, h_t$ $(t \geq 2)$ be nonzero meromorphic functions on $\mathbb{C}^m$ and A be a subset of $(1, +\infty)$ with infinite Lebesgue measure. Assume that

a) $h_0 + \cdots + h_t \equiv 0$,

b) $\sum_{v=0}^{t} N^{[1]}_{h_v}(r) + \sum_{v=0}^{t} N^{[1]}_{\frac{1}{h_v}}(r) \leq \frac{1}{t(t+1)} T_{\varphi_{ijk}}(r)$, $r \in A$ for all $\{i, j, k\} \subset \{0, 1, \dots, t\}$ such that $\frac{h_i}{h_j}, \frac{h_j}{h_k}, \frac{h_k}{h_i}$ are all nonconstant, where $\varphi_{ijk} := [h_i : h_j : h_k]$ is a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^2$.

Then there exists a decomposition of indices $\{0, \dots, t\} = I_1 \cup \cdots \cup I_s$ such that:

i) $\# I_v \geq 2$ for all $v \in \{1, \dots, s\}$,

ii) $i, j \in I_v$ if and only if $\frac{h_i}{h_j}$ is constant,

iii) $\sum_{j \in I_v} h_j = 0$, $v \in \{1, \dots, s\}$.

Proof. We prove this Lemma by induction on t.

+) If $t = 2$, we have

$$h_0 + h_1 + h_2 \equiv 0. \tag{1}$$

Case 1. If one of the meromorphic functions $\frac{h_0}{h_1}, \frac{h_1}{h_2}, \frac{h_2}{h_0}$ is constant, then by (1), we have that $h_0 : h_1 : h_2$ are constant. We get i), ii) and iii).

Case 2. If $\frac{h_0}{h_1}, \frac{h_1}{h_2}, \frac{h_2}{h_0}$ are nonconstant, by Theorem 5.2.29 in [7], we have

$$T_{\varphi_{012}}(r) = T_{[h_0:h_1:h_2]}(r) \leq T_{[h_0:h_1]}(r) + T_{[h_0:h_2]}(r) + 0(1).$$

Without loss of generality, we may assume that $T_{[h_0:h_1]}(r) \geq T_{[h_0:h_2]}(r)$ for all $r \in A_1$, where A_1 is a subset of A with infinite Lebesgue measure. Then

$$\sum_{i=0}^{2} N^{[1]}_{h_i}(r) + \sum_{i=0}^{2} N^{[1]}_{\frac{1}{h_i}}(r) \leq \frac{1}{6} T_{\varphi_{012}}(r) \leq \frac{1}{3} T_{[h_0:h_1]}(r), \quad r \in A_1.$$

Let $[h'_0 : h'_1]$ be a reduced representation of $[h_0 : h_1] : \mathbb{C}^m \longrightarrow \mathbb{C}P^1$, h'_0 and h'_1 are holomorphic functions. Set $h'_2 = \dfrac{h'_0 h_2}{h_0}$, then

$$h'_0 + h'_1 + h'_2 \equiv 0.$$

For each $j \in \{0, 1, 2\}$, we have that a zero of h'_j is a pole or a zero of some h_i $(i \in \{0, 1, 2\})$. On the other hand

$$\dim\{z : h'_0(z) = h'_1(z) = 0\} \leq m - 2.$$

Hence, we get

$$\sum_{i=0}^{2} N^{[1]}_{h'_i}(r) \leq 2 \cdot \Big(\sum_{i=0}^{2} N^{[1]}_{h_i}(r) + \sum_{i=0}^{2} N^{[1]}_{\frac{1}{h_i}}(r) \Big) \leq \frac{2}{3} T_{[h_0:h_1]}(r), \quad r \in A_1.$$

By the Second Main Theorem, we have:

$$\begin{aligned} T_{[h_0:h_1]}(r) &\leq N^{[1]}_{h'_0}(r) + N^{[1]}_{h'_1}(r) + N^{[1]}_{h'_0+h'_1}(r) + \circ\big(T_{[h_0:h_1]}(r)\big) \\ &= \sum_{i=0}^{2} N^{[1]}_{h'_i}(r) + \circ\big(T_{[h_0:h_1]}(r)\big) \\ &\leq \frac{2}{3} T_{[h_0:h_1]}(r) + \circ\big(T_{[h_0:h_1]}(r)\big), \quad r \in A_1. \end{aligned}$$

This is a contradiction when $r \to \infty$, $r \in A_1$.

This completes the proof of the case $t = 2$.

+) Assume that our assertion holds up to t $(t \geq 2)$. Consider

$$h_0 + \cdots + h_{t+1} \equiv 0. \tag{2}$$

We introduce an equivalence relation in $\{0, \ldots, t+1\}$ as follows: $i \sim j$ if and only if $\dfrac{h_i}{h_j}$ is constant. Let

$$\{I_1, \ldots, I_s\} = \{0, \ldots, t+1\}/\sim.$$

By definition we have ii).

For the proof of i), we assume that there exists I_v containing only one index, say $I_s = \{t+1\}$. Then $\dfrac{h_i}{h_{t+1}}$ $(i = 0, \ldots, t)$ are all nonconstant.

If $s = 2$ then $I_1 = \{0, \ldots, t\}$, $I_2 = \{t+1\}$.

By (2) we have

$$c \cdot h_0 + h_{t+1} \equiv 0, \quad c \in \mathbb{C}^*.$$

Thus $\dfrac{h_0}{h_{t+1}}$ is constant, this is a contradiction.

If $s = 3$, without loss of generality we may assume that $0 \in I_1$, $1 \in I_2$. By (2) we have

$$c \cdot h_0 + d \cdot h_1 + h_{t+1} \equiv 0, \quad c, d \in \mathbb{C} .$$

* If $c \cdot d = 0$, then $t+1 \in I_1$ or $t+1 \in I_2$, this is a contradiction.

* If $c \neq 0$, $d \neq 0$, we have:

$$T_{[c \cdot h_0 : d \cdot h_1 : h_{t+1}]}(r) = T_{[h_0 : h_1 : h_{t+1}]}(r) + 0(1) .$$

So by the basic step of induction, we have that $h_0 : h_1 : h_{t+1}$ are constant. This is a contradiction.

If $s > 3$, let $\Psi := [h_0 : \cdots : h_t] : \mathbb{C}^m \longrightarrow \mathbb{C}P^t$.

Let $[h'_0 : \cdots : h'_t]$ be a reduced representation of Ψ.

Set $h'_{t+1} = \dfrac{h'_t \cdot h_{t+1}}{h_t}$, then $h'_0 + \cdots + h'_{t+1} \equiv 0$.

For each $j \in \{0, \ldots, t+1\}$, we have that a zero of h'_j is a pole or a zero of some h_i $(i \in \{0, \ldots, t+1\})$.

Hence, we get

$$\begin{aligned} N^{[1]}_{h'_j}(r) &\leq \sum_{i=0}^{t+1} N^{[1]}_{h_i}(r) + \sum_{i=0}^{t+1} N^{[1]}_{\frac{1}{h_i}}(r) \\ &\leq \frac{1}{(t+1)(t+2)} T_{\varphi_{kpq}}(r) \\ &\leq \frac{1}{(t+1)(t+2)} T_{\Psi}(r), \quad r \in A \end{aligned}$$

where $k \in I_1$, $p \in I_2$, $q \in I_3$.

If Ψ is linearly nondegenerate, by the Second Main Theorem we have:

$$\begin{aligned} T_\Psi(r) &\le \sum_{i=0}^{t} N^{[t]}_{h'_i}(r) + N^{[t]}_{h'_0+\cdots+h'_t}(r) + \circ\big(T_\Psi(r)\big) \\ &= \sum_{t=0}^{t+1} N^{[t]}_{h'_i}(r) + \circ\big(T_\Psi(r)\big) \le t \cdot \sum_{i=0}^{t+1} N^{[1]}_{h'_i}(r) + \circ\big(T_\Psi(r)\big) \\ &\le \frac{t(t+2)}{(t+1)(t+2)} T_\Psi(r) + \circ\big(T_\Psi(r)\big) \\ &= \frac{t}{t+1} T_\Psi(r) + \circ\big(T_\Psi(r)\big), \quad r \in A\ . \end{aligned}$$

This is a contradiction when $r \to \infty$, $r \in A$.
Thus, Ψ is linearly degenerate, so there exist constants $(C_0, \ldots, C_t) \ne (0, \ldots, 0)$ such that

$$C_0 h_0 + \cdots + C_t h_t = 0\ . \tag{3}$$

We may assume that $C_0 = 1$. By (2) and (3) we have

$$(C_1 - 1)h_1 + \cdots + (C_t - 1)h_t - h_{t+1} \equiv 0\ .$$

It can be written in the form:

$$a_1 h_{i_1} + \cdots + a_k h_{i_k} + a_{t+1} h_{t+1} \equiv 0 \tag{4}$$

such that $a_i \in \mathbb{C}^*$, $a_{t+1} = -1$, $\dfrac{h_p}{h_q}$ is nonconstant for all $p \ne q \in \{i_1, \ldots, i_k, t+1\}$ and $k \le t - 1$.

+) If $k = 1$, by (4) we have that $h_{i_1} : h_{t+1}$ is constant. This is a contradiction.

+) If $k \ge 2$, for $\{p, q, v\} \subset \{i_1, \ldots, i_k, t+1\}$ we have

$$T_{[a_p h_p : a_q h_q : a_v h_v]}(r) = T_{[h_p : h_q : h_v]}(r) + 0(1).$$

By the induction hypothesis (since $k + 1 \le t$) there exists $p \in \{i_1, \ldots, i_k\}$ such that $a_p h_p : a_{t+1} h_{t+1}$ is constant. Thus $h_p : h_{t+1}$ is constant, this is a contradiction.

So $\# I_v \ge 2$ for all $v \in \{1, \ldots, s\}$, we get i).

Finally we show iii). We choose an index $v \in I_v$ and set

$$\sum_{i \in I_v} h_i = c_v \cdot h_v, \quad c_v \in \mathbb{C} \ .$$

Then (2) can be written as

$$\sum_{v=1}^{s} c_v \cdot h_v \equiv 0$$

By i) and the induction hypothesis, we infer like above that $c_v \equiv 0$. This shows iii). We have completed the proof of Lemma 3.1. □

We give the Second Main Theorem of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ with $(n+2)$ moving targets.

Lemma 3.2. *Let $f, g : \mathbb{C}^m \longrightarrow \mathbb{C}P^n$ be nonconstant meromorphic mappings and $\{a_j\}_{j=1}^{n+2}$ be "small" (with respect to g) meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ in general position.*

a) Denote the meromorphic mapping,

$$F = \big(c_1 \cdot (f, \tilde{a}_1) : \cdots : c_{n+1} \cdot (f, \tilde{a}_{n+1})\big) : \mathbb{C}^m \longrightarrow \mathbb{C}P^n$$

where $\{c_i\}_{i=1}^{n+1}$ are "small" (with respect to g) nonzero meromorphic functions on $\mathbb{C}^m$. Then we have

$$T_F(r) = T_f(r) + o(T_g(r)).$$

Moreover, if

$$\begin{aligned} f &= (f_1 : \cdots : f_{n+1}), \\ a_i &= (a_{i1} : \cdots : a_{i(n+1)}), \\ F &= \Big(\frac{c_1 \cdot (f, \tilde{a}_1)}{h} : \cdots : \frac{c_{n+1} \cdot (f, \tilde{a}_{n+1})}{h}\Big) \end{aligned}$$

are reduced representations, where h is a meromorphic function on $\mathbb{C}^m$, then

$$N_h(r) \leq o(T_g(r))$$

and

$$N_{\frac{1}{h}}(r) \leq o(T_g(r)).$$

b) *Assume that f is linearly nondegenerate over $\mathcal{R}\Big(\{a_j\}_{j=1}^{n+2}\Big)$. Then we have*

$$T_f(r) \leq \sum_{j=1}^{n+2} N^{[n]}_{(f,a_j)}(r) + o(T_f(r)) + o(T_g(r))$$

for all r except for a set of finite Lebesgue measure.

Proof.

a) Set

$$F_i = \frac{c_i \cdot (f, \tilde{a}_i)}{h}, \quad i \in \{1, \ldots, n+1\}.$$

So we have

$$\begin{cases} c_1 a_{10} f_0 + \cdots + c_1 a_{1n} f_n = h \cdot F_1 \cdot a_{1t_1} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ c_{n+1} a_{(n+1)0} f_0 + \cdots + c_{n+1} a_{(n+1)n} f_n = h \cdot F_{n+1} \cdot a_{(n+1)t_{n+1}} \end{cases} \tag{5}$$

Since $(F_1 : \cdots : F_{n+1})$ is a reduced representation of F, $\operatorname{codim}\{F_1 = \cdots = F_{n+1} = 0\} \geq 2$. Hence, by (5) we see:

$$N_{\frac{1}{h}}(r) \leq \sum_{i=1}^{n+1} N_{a_{it_i}}(r) + \sum_{i=1}^{n+1} N_{\frac{1}{c_i}}(r) = o(T_g(r)).$$

Set

$$P := \begin{pmatrix} c_1 a_{10} & \ldots & c_1 a_{1n} \\ \vdots & \ddots & \vdots \\ c_{n+1} a_{(n+1)0} & \ldots & c_{n+1} a_{(n+1)n} \end{pmatrix},$$

and matrices P_i ($i \in \{1, \ldots, n+1\}$) which are defined from P after changing the i^{th} column by $\begin{pmatrix} F_1 a_{1t_1} \\ \vdots \\ F_{(n+1)} a_{(n+1)t_{n+1}} \end{pmatrix}$.

Put $u = \det(P)$ and $u_i = \det(P_i)$, $i \in \{1, \dots, n+1\}$. It is easy to see that:

$$\frac{u}{a_{1t_1} \cdots a_{(n+1)t_{n+1}}} \in \mathcal{R}\big(\{a_j\}_{j=1}^{n+2}\big)$$

and

$$N_{\frac{1}{u_i}}(r) \leq 0\Big(\sum_{j=1}^{n+1} N_{\frac{1}{c_j}}(r)\Big) = o(T_g(r)), \quad i = 1, \dots, n+1.$$

By (5) we have

$$\begin{cases} f_0 = \dfrac{h \cdot u_1}{u} \\ \quad \cdots \\ f_n = \dfrac{h \cdot u_{n+1}}{u} \end{cases} \tag{6}$$

On the other hand $(f_0 : \cdots : f_n)$ is a reduced representation of f. Hence,

$$\begin{aligned} N_h(r) &\leq N_u(r) + \sum_{i=1}^{n+1} N_{\frac{1}{u_i}}(r) \\ &\leq N_{\frac{u}{a_{1t_1} \cdots a_{(n+1)t_{n+1}}}}(r) + N_{a_{1t_1} \cdots a_{(n+1)t_{n+1}}}(r) + \sum_{i=1}^{n+1} N_{\frac{1}{u_i}}(r) \\ &= o(T_g(r)). \end{aligned}$$

We have

$$
\begin{aligned}
T_F(r) &= \int\limits_{S(r)} \log\Big(\sum_{i=1}^{n+1} |F_i|^2\Big)^{1/2}\sigma + 0(1) \\
&= \int\limits_{S(r)} \log\Big(\sum_{i=1}^{n+1} \Big|\frac{c_i \cdot (f, \tilde{a}_i)}{h}\Big|^2\Big)^{1/2}\sigma + 0(1) \\
&= \int\limits_{S(r)} \log\Big(\sum_{i=1}^{n+1} |c_i(f, \tilde{a}_i)|^2\Big)^{1/2}\sigma - \int\limits_{S(r)} \log|h|\sigma + 0(1) \\
&\leq \int\limits_{S(r)} \log\|f\|\sigma + \int\limits_{S(r)} \log\left(\sum_{i=1}^{n+1} |c_i|^2\Big(\Big|\frac{a_{i0}}{a_{it_i}}\Big|^2 + \cdots + \Big|\frac{a_{in}}{a_{it_i}}\Big|^2\Big)\right)^{1/2}\sigma \\
&\quad - N_h(r) + N_{\frac{1}{h}}(r) + 0(1) \\
&\leq T_f(r) + \int\limits_{S(r)} \log^+\left(\sum_{i=1}^{n+1}\Big(\Big|c_i \cdot \frac{a_{i0}}{a_{it_i}}\Big|^2 + \cdots + \Big|c_i \cdot \frac{a_{in}}{a_{it_i}}\Big|^2\Big)\right)^{1/2}\sigma + o(T_g(r)) \\
&\leq T_f(r) + \sum_{i=1}^{n+1}\sum_{j=0}^{n} m\Big(r, c_i\frac{a_{ij}}{a_{it_i}}\Big) + o(T_g(r)) \\
&= T_f(r) + o(T_g(r)) \qquad (7)
\end{aligned}
$$

(note that $c_i \cdot \dfrac{a_{ij}}{a_{it_i}} \in \mathcal{R}\big(\{a_j\}_{j=1}^{n+2}\big)$).
(6) can be written as

$$
\begin{cases}
f_0 &= h \cdot \sum\limits_{j=1}^{n+1} b_{j0}F_j \\
\cdots & \quad \cdots \quad \cdots \\
f_n &= h \cdot \sum\limits_{j=1}^{n+1} b_{jn}F_j
\end{cases}
$$

where $b_{ji} \in \mathcal{R}\big(\{a_j\}_{j=1}^{n+2}\big)$.

So we get

$$
\begin{aligned}
T_f(r) &= \int\limits_{S(r)} \log\|f\|\sigma + 0(1) \\
&= \int\limits_{S(r)} \log\Big(\sum_{i=0}^{n} \Big|\sum_{j=1}^{n+1} b_{ji}F_j\Big|^2\Big)^{1/2}\sigma + \int\limits_{S(r)} \log|h|\sigma + 0(1) \\
&\le \int\limits_{S(r)} \log\|F\|\sigma + \int\limits_{S(r)} \log\Big(\sum_{i,j} |b_{ji}|^2\Big)^{1/2}\sigma + N_h(r) - N_{\frac{1}{h}}(r) + 0(1) \\
&\le T_F(r) + \int\limits_{S(r)} \log^+\Big(\sum_{i,j} |b_{ji}|^2\Big)^{1/2}\sigma + o(T_g(r)) \\
&\le T_F(r) + \sum_{i,j} m(r, b_{ij}) + o(T_g(r)) \\
&= T_F(r) + o(T_g(r)). \qquad (8)
\end{aligned}
$$

By (7) and (8), we have

$$T_F(r) = T_f(r) + o(T_g(r)).$$

This finishes the proof of part a).

b) We use a) for a special set of c_i : Set

$$N_{n+2} := \begin{pmatrix} a_{10} & \dots & a_{(n+1)0} \\ a_{11} & \dots & a_{(n+1)1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{(n+1)n} \end{pmatrix}$$

and matrices $N_i,\ i \in \{1, \dots, n+1\}$, which are defined from N_{n+2} after changing the i^{th} column by $\begin{pmatrix} a_{(n+2)0} \\ \vdots \\ a_{(n+2)n} \end{pmatrix}$.

Set

$$c_i = \frac{a_{it_i}}{a_{1t_1} \cdots a_{(n+2)t_{n+2}}} \cdot \det(N_i), \quad i \in \{1, \dots, n+2\},$$

then

$$c_i \in \mathcal{R}\big(\{a_j\}_{j=1}^{n+2}\big), \quad i \in \{1, \dots, n+2\}.$$

It is easy to see that:

$$\sum_{i=1}^{n+1} c_i \cdot (f, \tilde{a}_i) = c_{n+2} \cdot (f, \tilde{a}_{n+2}) \tag{9}$$

F is a linearly nondegenerate meromorphic mapping, since f is linearly nondegenerate over $\mathcal{R}\big(\{a_j\}_{j=1}^{n+2}\big)$ and since the a_j $(j = 1, \dots, n+2)$ are in general position.

Thus, by the First and the Second Main Theorem, we have

$$\begin{aligned}
T_f(r) + o(T_g(r)) = T_F(r) &\leq \sum_{j=1}^{n+1} N^{[n]}_{F_j}(r) + N^{[n]}_{F_1+\cdots+F_{n+1}}(r) + o(T_F(r)) \\
&\overset{(9)}{=} \sum_{j=1}^{n+2} N^{[n]}_{\frac{c_j\cdot(f,\tilde{a}_j)}{h}}(r) + o(T_f(r)) + o(T_g(r)) \\
&\leq \sum_{j=1}^{n+2} N^{[n]}_{(f,\tilde{a}_j)}(r) + \sum_{j=1}^{n+2} N_{c_j}(r) + (n+2)N_{\frac{1}{h}}(r) + o(T_f(r)) + o(T_g(r)) \\
&= \sum_{j=1}^{n+2} N^{[n]}_{(f,\tilde{a}_j)}(r) + o(T_f(r)) + o(T_g(r)) \\
&= \sum_{j=1}^{n+2} N^{[n]}_{(f,a_j)}(r) + o(T_f(r)) + o(T_g(r)).
\end{aligned}$$

This completes proof of Lemma 3.2. □

Proof of Theorem 1. Without loss of generality, we may assume that there exists a subset A of $(1, +\infty)$ with infinite Lebesgue measure such that

$$T_f(r) \geq T_g(r), \quad r \in A, \tag{10}$$

(note that if $T_g(r) \geq T_f(r)$ for all r except for a set of finite Lebesgue measure then $\{a_j\}_{j=1}^{3n+1}$ are "small" with respect to g). Define functions

$$h_j = \frac{(f, a_j)}{(g, a_j)}, \quad j \in \{1, \dots, 3n+1\}.$$

We choose an arbitrary subset $Q=\{j_1,\dots,j_{2n+2}\}$ of the index set $Q_0:=\{1,\dots,3n+1\}$.

We now prove that:

For each $I\subset Q$, $\#I=n+1$, there exists some $J\subset Q$ with $I\neq J$, $\#J=n+1$ such that

$$\frac{h_I}{h_J}\in\mathcal{R}\Big(\{a_j\}_{j=1}^{3n+1}\Big),\quad \text{where}\quad h_I=\prod_{i\in I}h_i. \tag{11}$$

We have

$$\begin{cases} a_{j0}f_0+\cdots+a_{jn}f_n=h_j(a_{j0}g_0+\cdots+a_{jn}g_n)\\ j\in Q\end{cases}$$

$$\Rightarrow\begin{cases} a_{j_s0}f_0+\cdots+a_{j_sn}f_n-h_{j_s}a_{j_s0}g_0-\cdots-h_{j_s}a_{j_sn}g_n=0\\ 1\le s\le 2n+2\end{cases}$$

Therefore, we get

$$\det(a_{j_s0},\dots,a_{j_sn},h_{j_s}a_{j_s0},\dots,h_{j_s}a_{j_sn},\ 1\le s\le 2n+2)\equiv 0\ .$$

For each $I=\{j_{s_0},\dots,j_{s_n}\}\subset Q$, $1\le s_0<\cdots<s_n\le 2n+2$, we define

$$A_I=\frac{(-1)^{\frac{n(n+1)}{2}+s_0+\cdots+s_n}\cdot\det(a_{j_{s_k}i},0\le k,i\le n)\cdot\det(a_{j_{s'_k}i},0\le k,i\le n)}{a_{j_1t_{j_1}}\dots a_{j_{2n+2}t_{j_{2n+2}}}}$$

where $\{s'_0,\dots,s'_n\}=\{1,\dots,2n+2\}\setminus\{s_0,\dots,s_n\}$, $s'_0<\cdots<s'_n$. We have $A_I\in\mathcal{R}\Big(\{a_j\}_{j=1}^{2n+2}\Big)$ and $A_I\not\equiv 0$, since $\{a_j\}_{j=1}^{2n+2}$ are in general position.

Set $L=\{I\subset Q,\#I=n+1\}$, then $\#L=N:=\begin{pmatrix}2n+2\\ n+1\end{pmatrix}$.

By the Laplace expansion Theorem, we have

$$\sum_{I\in L}A_Ih_I\equiv 0. \tag{12}$$

Let I, J, K be distinct in L . It is easy to see that:

$((I\cup J)\diagdown(I\cap J))\cap((J\cup K)\diagdown(J\cap K))\cap((K\cup I)\diagdown(K\cap I))=\phi$.

So, $C_{IJ}\cup C_{JK}\cup C_{KI}=\{1,...,n{+}2\}$, where $C_{IJ}=\{1,...,n{+}2\}\diagdown((I\cup J)\diagdown(I\cap J))$. Since $\dim\big({}^*E^i\cap{}^*E^j\big)\le m-2$ for all $i\neq j$, $i\in\{1,\dots,n+3\}$, $j\in\{1,\dots,3n+1\}$, ${}^*E^i\in\big\{{}^*E^i_f,{}^*E^i_g\big\}$, ${}^*E^j\in\big\{{}^*E^j_f,{}^*E^j_g\big\}$, and $f=g$ on

$\bigcup_{i=1}^{n+2} \left({}^*E_f^i \cap {}^*E_g^i\right)$ (note that in the case $n = 1$ we also have $f = g$ on $\bigcup_{i=1}^{3} \left({}^*E_f^i \cap {}^*E_g^i\right)$), we have :

$$N^{[1]}_{\frac{h_I}{h_J}-1}(r) + N^{[1]}_{\frac{h_J}{h_K}-1}(r) + N^{[1]}_{\frac{h_K}{h_I}-1}(r) + \sum_{j\in Q} {}^{>M}N^{[1]}_{(f,a_j)}(r) + \sum_{k=1}^{3n+1} {}^{>M}N^{[1]}_{(g,a_k)}(r)$$

$$\geq \sum_{i=1}^{n+2} {}^{\leq M}N^{[1]}_{(f,a_i)}(r) \tag{13}$$

Indeed, for $i \in \{1, ..., n+2\}$, we may assume that $i \in C_{IJ}$. Let $z_0 \in {}^*E_f^i$. If z_0 is not taken into account by $\sum_{j\in Q} {}^{>M}N^{[1]}_{(f,a_j)}(r)$ or by $\sum_{k=1}^{3n+1} {}^{>M}N^{[1]}_{(g,a_k)}(r)$ (this means that $v_{(f,a_j)}(z_0) \leq M$ and $v_{(g,a_k)}(z_0) \leq M$ for all $j \in Q$, $k \in \{1, ..., 3n+1\}$) then $z_0 \in {}^*E_g^i$ and by omitting an analytic set of codimension ≥ 2, we may assume that $(f, a_j)(z_0) \neq 0$ and $(g, a_k)(z_0) \neq 0$ for all $j \in Q \setminus \{i\}$, $k \in \{1, ..., 3n+1\} \setminus \{i\}$. In particular $(f, a_j)(z_0) \neq 0$, $(g, a_j)(z_0) \neq 0$ for all $j \in (I \cup J) \setminus (I \cap J)$.
On the other hand, $f(z_0) = g(z_0)$. Hence, $\frac{h_I}{h_J}(z_0) = 1$, this means that z_0 is taken into account by $N^{[1]}_{\frac{h_I}{h_J}-1}(r)$, so we get (13).
By Lemma 3.2 and the First Main Theorem, we have :

$$\begin{aligned} T_f(r) &\leq \sum_{i=1}^{n+2} N^{[n]}_{(f,a_i)}(r) + o\left(T_f(r)\right) \\ &\leq \frac{M}{M+1} \sum_{i=1}^{n+2} {}^{\leq M}N^{[n]}_{(f,a_i)}(r) + \frac{n}{M+1} \sum_{i=1}^{n+2} N_{(f,a_i)}(r) + o\left(T_f(r)\right) \\ &\leq \frac{Mn}{M+1} \sum_{i=1}^{n+2} {}^{\leq M}N^{[1]}_{(f,a_i)}(r) + \frac{n(n+2)}{M+1} T_f(r) + o\left(T_f(r)\right). \end{aligned}$$

Thus, we have

$$\frac{M+1-n(n+2)}{nM} T_f(r) \leq \sum_{i=1}^{n+2} {}^{\leq M}N^{[1]}_{(f,a_i)}(r) + \circ\big(T_f(r)\big). \tag{14}$$

By (13) and (14) we have :

$$N^{[1]}_{\frac{h_I}{h_J}-1}(r)+N^{[1]}_{\frac{h_J}{h_K}-1}(r)+N^{[1]}_{\frac{h_K}{h_I}-1}(r) \geq \frac{M+1-n(n+2)}{nM}T_f(r)-\sum_{j\in Q}{}^{>M}N^{[1]}_{(f,a_j)}(r)$$

$$-\sum_{k=1}^{3n+1}{}^{>M}N^{[1]}_{(g,a_k)}(r) - o\left(T_f(r)\right). \qquad (15)$$

We introduce an equivalence relation on L: $I \backsim J$ if and only if $\frac{h_I}{h_J} \in \mathcal{R}\Big(\big\{a_j\big\}_{j=1}^{3n+1}\Big)$.

Set $\{L_1, ..., L_s\} = L \diagup \backsim$, ($s \leq N := \begin{pmatrix} 2n+2 \\ n+1 \end{pmatrix}$).

In order to prove (11), we show that $\# L_v \geq 2$ for all $v \in \{1, \ldots, s\}$. For each $v \in \{1, \ldots, s\}$, choose $I_v \in L_v$ and set

$$\sum_{I \in_v} A_I h_I = B_v h_{I_v}, \quad B_v \in \mathcal{R}\Big(\big\{a_j\big\}_{j=1}^{3n+1}\Big).$$

Then (12) can be written as

$$\sum_{v=1}^{s} B_v h_{I_v} \equiv 0. \qquad (16)$$

+) If $B_v \equiv 0$ for all $v \in \{1, \ldots, s\}$, then $\# L_v \geq 2$ for all $v \in \{1, \ldots, s\}$ by $A_I \not\equiv 0$, $I \in L$. We get (11).

+) If there exists some $B_v \not\equiv 0$, then by (16) there are at least 3 of the $B_1, \ldots, B_s$ different from zero since $h_I \not\equiv 0$, $\dfrac{h_{I_i}}{h_{I_j}} \notin \mathcal{R}\Big(\big\{a_j\big\}_{j=1}^{3n+1}\Big)$, $(1 \leq i \neq j \leq s$, $I \in L$).

We want to apply Lemma 3.1 to (16), without loss of generality we may assume that $B_v \not\equiv 0$ for all $v \in \{1, \ldots, s\}$.

For each $\{i, j, k\} \subset \{1, \ldots, s\}$, set

$$T(r) = T_{\frac{B_i}{B_j}}(r) + T_{\frac{B_j}{B_k}}(r) + T_{\frac{B_k}{B_i}}(r)$$

then $T(r) = \circ\big(T_f(r)\big)$ as $r \to \infty$.

It is clear that $\dfrac{h_{I_i}}{h_{I_j}} - 1 \not\equiv 0$, $\dfrac{h_{I_j}}{h_{I_k}} - 1 \not\equiv 0$, $\dfrac{h_{I_k}}{h_{I_i}} - 1 \not\equiv 0$.

By (10), (15), Theorem 5.2.29 in [7] and the First Main Theorem, we have:

$$3 \cdot T_{[B_i h_{I_i} : B_j h_{I_j} : B_k h_{I_k}]}(r) + 0(1) \geq T_{\frac{B_i h_{I_i}}{B_j h_{I_j}}}(r) + T_{\frac{B_j h_{I_j}}{B_k h_{I_k}}}(r) + T_{\frac{B_k h_{I_k}}{B_i h_{I_i}}}(r)$$
$$\geq T_{\frac{h_{I_i}}{h_{I_j}}}(r) + T_{\frac{h_{I_j}}{h_{I_k}}}(r) + T_{\frac{h_{I_k}}{h_{I_i}}}(r) - T(r)$$
$$\geq N_{\frac{h_{I_i}}{h_{i_j}} - 1}(r) + N_{\frac{h_{I_j}}{h_{I_k}} - 1}(r) + N_{\frac{h_{I_k}}{h_{I_i}} - 1}(r) - \circ\big(T_f(r)\big)$$
$$\overset{(15)}{\geq} \frac{M + 1 - n(n+2)}{nM} T_f(r) - \sum_{j \in Q} {}^{>M} N^{[1]}_{(f,a_j)}(r) - \sum_{j=1}^{3n+1} {}^{>M} N^{[1]}_{(g,a_j)}(r) - \circ\big(T_f(r)\big)$$
$$\geq \frac{M + 1 - n(n+2)}{nM} T_f(r) - \frac{1}{M+1} \sum_{j \in Q} N_{(f,a_j)}(r) - \frac{1}{M+1} \sum_{j=1}^{3n+1} N_{(g,a_j)}(r) - \circ\big(T_f(r)\big)$$

$$\geq \frac{M + 1 - n(n+2)}{n \cdot M} T_f(r) - \frac{2(n+1)}{M+1} T_f(r) - \frac{3n+1}{M+1} T_g(r) - \circ\big(T_f(r)\big)$$
$$\geq \Big(\frac{M + 1 - n(n+2)}{nM} - \frac{5n+3}{M+1}\Big) T_f(r) - \circ\big(T_f(r)\big), \ r \in A. \tag{17}$$

Since $v_{(f,a_j)} = v_{(g,a_j)}$ on $E^j_f \cap E^j_g$, $j = 1, \ldots, 3n+1$, we have

$$\{z \in \mathbb{C}^m : h_I(z) = 0 \text{ or } h_I(z) = \infty\} \subset \bigcup_{j \in I} \{z \in \mathbb{C}^m : v_{(f,a_j)} > M \text{ or } v_{(g,a_j)}(z) > M\}$$

for all $I \subset \{1, \ldots, 3n+1\}$, $\#I = n+1$. Thus, we get

$$\begin{aligned} N^{[1]}_{h_I}(r) + N^{[1]}_{\frac{1}{h_I}}(r) &\leq \sum_{j \in I} {}^{>M} N^{[1]}_{(f,a_j)}(r) + \sum_{j \in I} {}^{>M} N^{[1]}_{(g,a_j)}(r) \\ &\leq \frac{1}{M+1} \Big(\sum_{j \in I} N_{(f,a_j)}(r) + \sum_{j \in I} N_{(g,a_j)}(r) \Big) \\ &\leq \frac{n+1}{M+1} \Big(T_f(r) + T_g(r) \Big) + 0(1) \\ &\overset{(10)}{\leq} \frac{2(n+1)}{M+1} T_f(r) + 0(1), \quad r \in A. \end{aligned}$$

So, we have

$$\sum_{v=1}^{s} N^{[1]}_{B_v h_{I_v}}(r) + \sum_{v=1}^{s} N^{[1]}_{\frac{1}{B_v h_{I_v}}}(r) \le \sum_{v=1}^{s} N^{[1]}_{h_{I_v}}(r) + \sum_{v=1}^{s} N^{[1]}_{\frac{1}{h_{I_v}}}(r) +$$
$$+ \sum_{v=1}^{s} N^{[1]}_{B_v}(r) + \sum_{v=1}^{s} N^{[1]}_{\frac{1}{B_v}}(r) \le \sum_{v=1}^{s} \Big(N^{[1]}_{h_{I_v}}(r) + N^{1]}_{\frac{1}{h_{I_v}}}(r) \Big) + \circ\big(T_f(r)\big)$$
$$\le \frac{2s(n+1)}{M+1} T_f(r) + \circ\big(T_f(r)\big) \le \frac{2(n+1)N}{M+1} T_f(r) + \circ\big(T_f(r)\big), \quad r \in A. \tag{18}$$

By (17), (18) we have

$$\sum_{v=1}^{s} N^{[1]}_{B_v h_v}(r) + \sum_{v=1}^{s} N^{[1]}_{\frac{1}{B_v h_v}}(r) \le \frac{6n(n+1)NM}{(M+1)^2 - n(n+2)(M+1) - n(5n+3)M} \cdot$$
$$\cdot T_{[B_i h_{I_i}:B_h h_{I_j}:B_k h_{I_k}]}(r) + \circ\big(T_{[B_i h_{I_i}:B_j h_{I_j}:B_k h_{I_k}]}(r)\big)$$
$$< \frac{1}{N(N-1)} T_{\varphi_{ijk}}(r) \le \frac{1}{s(s-1)} T_{\varphi_{ijk}}(r), \quad r \in A, \tag{19}$$

where $\varphi_{ijk} := [B_i h_{I_i} : B_j h_{I_j} : B_k h_{I_k}]$.

Then by applying Lemma 3.1 to (16) we get: For each $i \in \{1, \dots, s\}$ there exists $j \in \{1, \dots, s\}$, $j \ne i$ such that $\frac{B_i h_{I_i}}{B_j h_{I_j}}$ is constant.

So $\frac{h_{I_i}}{h_{I_j}} \in \mathcal{R}\Big(\big\{a_j\big\}_{j=1}^{3n+1}\Big)$, this means that $L_i \cap L_j \ne \emptyset$. This is a contradiction.

We have completed proof of (11). □

Let $\mathcal{M}^*$ be the abelian multiplication group of all nonzero meromorphic functions on $\mathbb{C}^m$. Define $\mathcal{H} \subset \mathcal{M}^*$ by the set of all $h \in \mathcal{M}^*$ with $h^k \in \mathcal{R}\Big(\big\{a_j\big\}_{j=1}^{3n+1}\Big)$ for some positive integer k. It is easy to see that $\mathcal{H}$ is a subgroup of $\mathcal{M}^*$.
We have

$$\mathcal{M}^* \cap \mathcal{R}\Big(\big\{a_j\big\}_{j=1}^{3n+1}\Big) \subset \mathcal{H} \subset \widetilde{\mathcal{R}}\Big(\big\{a_j\big\}_{j=1}^{3n+1}\Big),$$

and the multiplication group $\mathcal{G} := \mathcal{M}^* \diagup \mathcal{H}$ is a torsion free abebian group. We denote by $[h]$ the class in $\mathcal{G}$ containing $h \in \mathcal{M}^*$. Consider the subgroup $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ generated by $[h_1], \dots, [h_{3n+1}]$ and choose suitable functions $\eta_1, \dots, \eta_t \in \mathcal{M}^*$ such that $[\eta_1], \dots, [\eta_t]$ give a basis of $\widetilde{\mathcal{G}}$. Then each h_j can be uniquely represented as $h_j = c_j\eta_1^{\ell_{j1}} \cdots \eta_t^{\ell_{jt}}$, $c_j \in \mathcal{H}$, $\ell_{j_r} \in \mathbb{Z}$. For these integers ℓ_{j_r} we can choose suitable integers $p_1, \dots, p_t$ satisfying the condition: For integers $\ell_j = p_1\ell_{j_1} + \cdots + p_t\ell_{j_t}$, $(1 \le j \le 3n+1)$, $\ell_i = \ell_j$ if and only if $(\ell_{i_1}, \dots, \ell_{i_t}) = (\ell_{j_1}, \dots, \ell_{j_t})$, or equivalently

$$\frac{h_i}{h_j} \in \mathcal{H}.$$

We now show that:
There is a subset $I_0 = \{j_0, ..., j_n\} \subset Q_0$ such that

$$\frac{h_i}{h_j} \in \mathcal{H} \quad \text{for all} \quad i, j \in I_0. \tag{20}$$

We assume that, after a suitable change of indices, we have $\ell_1 \le \cdots \le \ell_{3n+1}$.

Take the subset $Q = \{1, \dots, n+1, 2n+1, \dots, 3n+1\}$ of Q_0 which contains $(2n+2)$ elements and apply (11) to the h'_j $(j \in Q)$ to show that there is a subset $\{i_0, \dots, i_n\}$ of Q satisfying the condition that $\{i_0, \dots, i_n\} \ne \{1, \dots, n+1\}$, $i_0 < \cdots < i_n$ and

$$\frac{h_{i_0} \cdots h_{i_n}}{h_1 \cdots h_{n+1}} \in \mathcal{R}\Big(\big\{a_j\big\}_{j=1}^{3n+1}\Big).$$

From this, it follows that

$$(\ell_{i_0} - \ell_1) + \cdots + (\ell_{i_n} - \ell_{n+1}) = \sum_{s=1}^{t} p_s(\ell_{i_{0_s}} + \cdots + \ell_{i_{n_s}} - \ell_{1_s} - \cdots - \ell_{n+1_s}) = 0.$$

Since $\ell_{i_0} \ge \ell_1, \dots, \ell_{i_n} \ge \ell_{n+1}$, this is posible only if $\ell_{n+1} = \ell_{i_n}$ so $\ell_{n+1} = \cdots = \ell_{2n+1}$ (note that $i_n \ge 2n+1$). Then take $I_0 = \{n+1, ..., 2n+1\}$, so we get (20). $\square$

Let $u_i = \dfrac{h_{j_i}}{h_{j_0}} \in \mathcal{H} \subset \widetilde{\mathcal{R}}\Big(\big\{a_j\big\}_{j=1}^{3n+1}\Big)$, $i \in \{0, \dots, n\}$. Then we have:

$$\begin{cases} a_{j_i0}f_0 + \cdots + a_{j_in}f_n = u_ih_{j_0}(a_{j_i0}g_0 + \cdots + a_{j_in}g_n) \\ \qquad\qquad i = 0, \dots, n. \end{cases} \tag{21}$$

+)Assume $n = 1$.

Set $A = \begin{pmatrix} a_{j_00} & a_{j_01} \\ a_{j_10} & a_{j_11} \end{pmatrix}, B = \begin{pmatrix} u_0 & 0 \\ 0 & u_1 \end{pmatrix}$.

By (21) , $A.\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = h_{j_0}.B.A.\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} \Rightarrow A^{-1}.B^{-1}.A.\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = h_{j_0}.\begin{pmatrix} g_0 \\ g_1 \end{pmatrix}$

We get 1) of the Theorem 1 (with $L = A^{-1}.B^{-1}.A$).

+) Assume $n \geq 2$.

Set $F = ((f, \widetilde{a_{j_0}}) : ... : (f, \widetilde{a_{j_n}}))$ and $G = ((g, \widetilde{a_{j_0}}) : ... : (g, \widetilde{a_{j_n}}))$. They are meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$. Take meromorphic functions h, u on $\mathbb{C}^m$ such that $F = \left(\frac{(f,\widetilde{a_{j_0}})}{h} : ... : \frac{(f,\widetilde{a_{j_n}})}{h}\right), G = \left(\frac{(g,\widetilde{a_{j_0}})}{u} : ... : \frac{(g,\widetilde{a_{j_n}})}{u}\right)$ are reduced representations. By Lemma 3.2 we have $N_h(r) = o\left(T_f(r)\right)$, $N_u(r) = o\left(T_f(r)\right)$, $N_{\frac{1}{h}}(r) = o(T_f(r))$ and $N_{\frac{1}{u}}(r) = o(T_f(r))$.

Put $F_i := \frac{(f,\widetilde{a_{j_i}})}{h}, G_i := \frac{(g,\widetilde{a_{j_i}})}{u}, i \in \{0, ..., n\}$.

Since $u_i \in \mathcal{H}$, $i \in \{0, \ldots, n\}$, we can choose a positive k such that $(u_i)^k \in \mathcal{R}\Big(\{a_j\}_{j=1}^{3n+1}\Big)$ for all $i \in \{0, ..., n)$.

By (21) we have

$$\begin{cases} F_i = \frac{u_i h_{j_0} u G_i}{h} \\ i = 0, ..., n \end{cases}$$

Since $G = (G_0 : ... : G_n)$ is a reduced representation and F_i $(i = 1, ..., n)$ are holomorphic functions , we have:

$$N_{\frac{1}{h_{j_0}}}(r) \leq \sum_{i=0}^{n} N_{u_i}(r) + N_u(r) + N_{\frac{1}{h}}(r) \leq \sum_{i=0}^{n} N_{(u_i)^k}(r) + N_u(r) + N_{\frac{1}{h}}(r)$$

$$\leq \sum_{i=0}^{n} T_{(u_i)^k}(r) + 0(1) + N_u(r) + N_{\frac{1}{h}}(r) = o\left(T_f(r)\right). \tag{22}$$

Suppose that $F \not\equiv G$, then there exist $0 \leq s < v \leq n$ such that :

$\begin{vmatrix} F_s & F_v \\ G_s & G_v \end{vmatrix} \not\equiv 0 \Rightarrow \left(\frac{h}{u h_{j_0} u_v} - \frac{h}{u h_{j_0} u_s}\right) F_s F_v \not\equiv 0$.

Define the meromorphic mapping $F \wedge G := (.... : \begin{vmatrix} F_i & F_j \\ G_i & G_j \end{vmatrix} : ...) : \mathbb{C}^m \longrightarrow \mathbb{C}P^{N_2}$, $(\ 0 \leq i < j \leq n \ , \ N_2 = \begin{pmatrix} n+1 \\ 2 \end{pmatrix} - 1 \ )$.

Take $\mu_{F\wedge G}$ a holomophic function on $\mathbb{C}^m$ such that $(.... : \frac{1}{\mu_{F\wedge G}} \begin{vmatrix} F_i & F_j \\ G_i & G_j \end{vmatrix} : ...)$ is a reduded representation of $F \wedge G$.

It is easy to see that there exists a subset $I_{sv} \subseteq \{1, ..., n+4\} \diagdown \{j_s, j_v\}$ such that

$$\# I_{sv} = n+2, \# \left(\{1, ..., n+3\} \diagdown (I_{sv} \cup \{j_s\}) \ \ \right) \leq 1, \text{ and}$$

$$\# \left(\{1, ..., n+3\} \diagdown (I_{sv} \cup \{j_v\}) \ \ \right) \leq 1. \tag{23}$$

In fact, we take $I_{sv} = \{1, ..., n+2\}$ if $\{j_s, j_v\} \cap \{1, ..., n+3\} = \phi$, $I_{sv} = \{1, ..., n+3\} \diagdown \{j_s, j_v\}$ if $\#(\{j_s, j_v\} \cap \{1, ..., n+3\}) = 1$ and
$I_{sv} = \{1, ..., n+4\} \diagdown \{j_s, j_v\}$ if $\{j_s, j_v\} \subset \{1, ..., n+3\}$.
By assumptions ii) and iii) we have :

$$N_{\frac{1}{F_s F_v}\mu_{F\wedge G}}(r) \geq \sum_{i\in I_{sv}} {}^{\leq M}N^{[1]}_{(f,a_i)}(r) - \sum_{j\in\{j_s,j_v\}} {}^{>M}N^{[1]}_{(f,a_j)}(r) - \sum_{i\in I_{sv}} {}^{>M}N^{[1]}_{(g,a_i)}(r) \tag{24}$$

Indeed, for $i_0 \in I_{sv}$, let $z_0 \in {}^*E_f^{i_0}$ be a generic point of a component D of ${}^*E_f^{i_0}$. If z_0 is not taken into account by $\sum_{j\in\{j_s,j_v\}} {}^{>M}N^{[1]}_{(f,a_j)}(r)$ or by $\sum_{i\in I_{sv}} {}^{>M}N^{[1]}_{(g,a_i)}(r)$ (this means that $v_{(f,a_j)}(z_0) \leq M$, $j \in \{j_s, j_v\}$ and $v_{(g,a_i)}(z_0) \leq M$, $i \in I_{sv}$) then $z_0 \in {}^*E_g^{i_0}$ (which implies $f(z_0) = g(z_0)$). Since $z_0 \in D$ is generic, we can omit an analytic set of codimension ≥ 2, so we may assume that $(f, a_{j_s})(z_0) \neq 0$,$(f, a_{j_v})(z_0) \neq 0$ (note that by (23) we cannot have $\{i_0, j_s\} \subset \{n+4, ..., 3n+1\}$ or $\{i_0, j_v\} \subset \{n+4, ..., 3n+1\}$), which implies $F_s(z_0) \neq 0$, $F_v(z_0) \neq 0$. Since we have $f(z_0) = g(z_0)$ on D, we get $\mu_{F\wedge G}(\ z_0) = 0$ on D. This means that z_0 is taken into account by $N_{\frac{1}{F_s F_v}\mu_{F\wedge G}}(r)$, so we get (24).
So, we have

$$\begin{aligned}
N_{\frac{1}{F_s F_v}\mu_{F\wedge G}}(r) &\geq \sum_{i\in I_{sv}} {}^{\leq M}N^{[1]}_{(f,a_i)}(r) - \frac{1}{M+1}\left(\sum_{j\in\{j_s,j_v\}} N_{(f,a_j)}(r) - \sum_{i\in I_{sv}} N_{(g,a_i)}(r)\right) \\
&\geq \sum_{i\in I_{sv}} {}^{\leq M}N^{[1]}_{(f,a_i)}(r) - \frac{2}{M+1}T_f(r) - \frac{n+2}{M+1}T_g(r) \\
&\geq \sum_{i\in I_{sv}} {}^{\leq M}N^{[1]}_{(f,a_j)}(r) - \frac{(n+4)}{M}T_f(r), r \in A.
\end{aligned} \tag{25}$$

By Lemma 3.2 and the First Main Theorem, we have:

$$
\begin{aligned}
T_f(r) &\leq \sum_{i\in I_{sv}} N^{[n]}_{(f,a_i)}(r) + o\left(T_f(r)\right)\\
&\leq \frac{M}{M+1}\sum_{i\in I_{sv}} {}^{\leq M}N^{[n]}_{(f,a_i)}(r) + \frac{n}{M+1}\sum_{i\in I_{sv}} N_{(f,a_i)}(r) + o\left(T_f(r)\right)\\
&\leq \frac{Mn}{M+1}\sum_{i\in I_{sv}} {}^{\leq M}N^{[1]}_{(f,a_i)}(r) + \frac{n(n+2)}{M+1}T_f(r) + o\left(T_f(r)\right).
\end{aligned}
$$

$$
\Rightarrow \left(\frac{M+1-n(n+2)}{Mn}\right) T_f(r) \leq \sum_{i\in I_{sv}} {}^{\leq M}N^{[1]}_{(f,a_i)}(r) + o\left(T_f(r)\right).
$$

So, by (25) we have:

$$
\left(\frac{M+1-2n(n+3)}{Mn}\right) T_f(r) \leq N_{\frac{1}{F_sF_v}\mu_{F\wedge G}}(r) + o\left(T_f(r)\right), r \in A. \qquad (26)
$$

By the definition of $\mu_{F\wedge G}$, we have:

$$
\begin{aligned}
N_{\frac{1}{F_sF_v}\mu_{F\wedge G}}(r) &\leq N_{\frac{1}{F_sF_v}\left|\begin{matrix} F_s & F_v \\ G_s & G_v\end{matrix}\right|}(r) = N_{\left(\frac{h}{uh_{j_0}u_v} - \frac{h}{uh_{j_0}u_s}\right)}(r) + N_{\frac{1}{u}}(r)\\
&\leq N_{\left(\frac{1}{u_v}-\frac{1}{u_s}\right)}(r) + N_{\frac{1}{h_{j_0}}}(r) + N_h(r) + N_{\frac{1}{u}}(r)\\
&\overset{(22)}{\leq} N_{\left(\frac{1}{(u_v)^k}-\frac{1}{(u_s)^k}\right)}(r) + o\left(T_f(r)\right) \leq T_{\left(\frac{1}{(u_v)^k}-\frac{1}{(u_s)^k}\right)}(r) + o\left(T_f(r)\right)\\
&\leq T_{\frac{1}{(u_v)^k}}(r) + T_{\frac{1}{(u_s)^k}}(r) + o\left(T_f(r)\right) = o\left(T_f(r)\right).
\end{aligned}
$$

This contracdicts to (26). Thus $F = G \Rightarrow f = g$, so we get 2) of Theorem 1. This completes the proof of Theorem 1. □

We can obtain Theorem 2 by an argument similar to the proof of Theorem 1 with the following remarks:

+) We do not need the assumption (10).

+) Similarly to (13) we have :

$$N^{[1]}_{\frac{h_I}{h_J}-1}(r) + N^{[1]}_{\frac{h_J}{h_K}-1}(r) + N^{[1]}_{\frac{h_K}{h_I}}(r) + \sum_{j\in Q}{}^{>M} N^{[1]}_{(f,a_j)}(r)$$

$$\geq \sum_{i=1}^{n+2}{}^{\leq M} N^{[1]}_{(f,a_i)}(r) \quad \text{for all } r.$$

So, similarly to (17) we have,

$$3\cdot T_{[B_i h_{I_i}:B_j h_{I_j}:B_k h_{I_k}]}(r) \geq \Big(\frac{(M+1)-n(n+2)}{nM} - \frac{2(n+1)}{M+1}\Big)T_f(r) + \circ\big(T_f(r)\big), \tag{17'}$$

+) Since $\min\big\{v_{(f,a_j)}, M\big\} = \min\big\{v_{(g,a_j)}, M\big\}$, $j\in\{1,\ldots,3n+1\}$, we have

$$\big\{z\in\mathbb{C}^m : h_I(z)=0 \text{ or } h_I(z)=\infty\big\} \subset \bigcup_{j\in I}\big\{z\in\mathbb{C}^m : v_{(f,a_j)}(z) > M\big\}$$

for all $I\subset\{1,\ldots,3n+1\}$, $\#I = n+1$.

Thus, we get

$$N^{[1]}_{h_I}(r) + N^{[1]}_{\frac{1}{h_I}}(r) \leq \sum_{j\in I}{}^{>M} N^{[1]}_{(f,a_j)}(r) \leq \frac{1}{M+1}\sum_{j\in I} N_{(f,a_j)}(r) \leq \frac{n+1}{M+1}T_f(r) + 0(1)$$

So, similarly to (18) we have,

$$\sum_{v=1}^{s} N^{[1]}_{B_v h_{I_v}}(r) + \sum_{v=1}^{s} N^{[1]}_{\frac{1}{B_v h_{I_v}}}(r) \leq \frac{N(n+1)}{M+1}T_f(r) + \circ\big(T_f(r)\big) \quad \text{for all } r. \tag{18'}$$

By (17') and (18') we have :

$$\sum_{v=1}^{s} N^{[1]}_{B_v h_{I_v}}(r) + \sum_{v=1}^{s} N^{[1]}_{\frac{1}{B_v h_{I_v}}}(r) \leq \frac{1}{s(s-1)}T_{\varphi_{ijk}}(r) \quad \text{for all } r.$$

+) Similarly to (24) we have:

$$N_{\frac{1}{F_s F_v}\mu_{F\wedge G}}(r) \geq \sum_{i\in I_{sv}}{}^{\leq M} N^{[1]}_{(f,a_i)}(r) - \sum_{j\in\{j_s,j_v\}}{}^{>M} N^{[1]}_{(f,a_j)}(r).$$

So, similarly to (25) we have:

$$N_{\frac{1}{F_s F_v}\mu_{F\wedge G}}(r) \geq \sum_{i\in I_{sv}}{}^{\leq M} N^{[1]}_{(f,a_i)}(r) - \frac{2}{M}T_f(r).$$

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