AN EXTENSION OF UNIQUENESS THEOREMS FOR MEROMORPHIC MAPPINGS

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Abstract

In this paper, we give some results on the number of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ under a condition on the inverse images of hyperplanes in $\mathbb{C}P^n$. At the same time, we give an answer for an open question posed by H. Fujimoto in 1998.

1 Introduction

In 1926, R. Nevanlinna showed that for two nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$, if they have the same inverse images for five distinct values, then $f = g$, and that g is a special type of a linear fractional tranformation of f if they have the same inverse images, counted with multiplicities, for four distinct values.

In 1975, H. Fujimoto [2] generalized Nevanlinna's result to the case of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$. This problem continued to be studied by L. Smiley [9], S.Ji [5] and others.

Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and H be a hyperplane in $\mathbb{C}P^n$ such that $\mathrm{im} f \not\subseteq H$. Denote by $v_{(f,H)}$ the map of $\mathbb{C}^m$ into $\mathbb{N}_0$ such that $v_{(f,H)}(a)$ $(a \in \mathbb{C}^m)$ is the intersection multiplicity of the image of f and H at $f(a)$. Let k be a positive interger or $+\infty$. We set

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$$v^{k)}_{(f,H)}(a) = \begin{cases} 0 & \text{if} \quad v_{(f,H)}(a) > k, \\ v_{(f,H)}(a) & \text{if} \quad v_{(f,H)}(a) \le k. \end{cases}$$

Let f be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $\{H_j\}_{j=1}^q$ be q hyperplanes in general position with

(a) $\quad dim \left\{z : v^{k)}_{(f,H_i)}(z) > 0 \quad \text{and} \quad v^{k)}_{(f,H_j)}(z) > 0\right\} \le m-2$ for all $1 \le i < j \le q$.

For each positive integer p, denote by $F_k(\{H_j\}_{j=1}^q, f, p\ )$ the set of all linearly nondegenerate meromorphic mappings g of $\mathbb{C}^m$ into $\mathbb{C}P^n$ such that:

(b) $\quad min\{v^{k)}_{(g,H_j)}, p\} = min\{v^{k)}_{(f,H_j)}, p\}$,

(c) $\quad g = f$ on $\bigcup\limits_{j=1}^{q} \{z : v^{k)}_{(f,H_j)}(z) > 0\}$.

In [5], S.Ji showed the following

Theorem J. ([5]) *If* $q = 3n+1$ *and* $k = +\infty$, *then for three mappings* $f_1, f_2, f_3 \in F_k\big(\{H_j\}_{j=1}^q, f, 1\big)$, *the mapping* $f_1 \times f_2 \times f_3 : \mathbb{C}^m \longrightarrow \mathbb{C}P^n \times \mathbb{C}P^n \times \mathbb{C}P^n$ *is algebraically degenerate, namely,* $\{(f_1(z), f_2(z), f_3(z)), z \in \mathbb{C}^m\}$ *is contained in a proper algebraic subset of* $\mathbb{C}P^n \times \mathbb{C}P^n \times \mathbb{C}P^n$.

In 1929, H. Cartan declared that there are at most two meromorphic functions on $\mathbb{C}$ which have the same inverse images (ignoring multiplicities) for four distinct values. However in 1988, N. Steinmetz ([10]) gave examples which showed that H. Cartan's declaration is false. On the other hand, in 1998, Fujimoto ([4]) showed that H. Cartan's declaration is true if we assume that meromorphic functions on $\mathbb{C}$ share four distinct values counted with multiplicities truncated by 2. He gave the following theorem

Theorem F. ([4]) *If* $q = 3n+1$ *and* $k = +\infty$ *then* $F_k\big(\{H_j\}_{j=1}^q, f, 2\big)$ *contains at most two mappings.*

He also proposed an open problem asking if the number $q = 3n+1$ in Theorem F can be replaced by a smaller one. Inspired by this question, in this paper we will generalize the above results to the case where the number $q = 3n+1$ is in fact replaced by a smaller one. We also obtain an improvement concerning truncating multiplicities.

Denote by Ψ the Segre embedding of $\mathbb{C}P^n \times \mathbb{C}P^n$ into $\mathbb{C}P^{n^2+2n}$ which is defined by sending the ordered pair $((w_0,...,w_n),(v_0,...,v_n))$ to $(...,w_iv_j,...)$ (in lexicographic order).

Let $h:\mathbb{C}^m \longrightarrow \mathbb{C}P^n\times\mathbb{C}P^n$ be a meromorphic mapping. Let $(h_0:...:h_{n^2+2n})$ be a representation of $\Psi\circ h$. We say that h is linearly degenerate (with the algebraic structure in $\mathbb{C}P^n\times\mathbb{C}P^n$ given by the Segre embedding) if $h_0,...,h_{n^2+2n}$ are linearly dependent over $\mathbb{C}$.

Our main results are stated as follows:

Theorem 1. *There are at most two distinct mappings in* $F_k\big(\{H_j\}_{j=1}^q,f,p\big)$ *in each of the following cases*:

i) $1\le n\le 3, q=3n+1, p=2$ *and* $23n\le k\le +\infty$

ii) $4\le n\le 6, q=3n, p=2$ *and* $\frac{(6n-1)n}{n-3}\le k\le+\infty$

iii) $n\ge 7, q=3n-1, p=1$ *and* $\frac{(6n-4)n}{n-6}\le k\le+\infty$.

Theorem 2. *Assume that* $q=\left[\frac{5(n+1)}{2}\right], (65n+171)n\le k\le+\infty$ *, where* $[x]:=\max\{d\in\mathbb{N}: d\le x\}$ *for a positive constant* x*. Then one of following assertions holds :*

i) $\#F_k\big(\{H_j\}_{j=1}^q,f,1\big)\le 2.$

ii) *For any* $f_1,f_2\in F_k\big(\{H_j\}_{j=1}^q,f,1\big)$*, the mapping* $f_1\times f_2:\mathbb{C}^m\longrightarrow \mathbb{C}P^n\times\mathbb{C}P^n$ *is linearly degenerate (with the algebraic structure in* $\mathbb{C}P^n\times\mathbb{C}P^n$ *given by the Segre embedding).*

We finally remark that we obtained similar uniqueness theorems with moving targets in [11], but only with a bigger number of targets and with much bigger truncations.

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2 Preliminaries

We set $\|z\|:=(|z_1|^2+\cdots+|z_m|^2)^{1/2}$ for $z=(z_1,\ldots,z_m)\in\mathbb{C}^m$, $B(r):=\big\{z:\|z\|<r\big\}$, $S(r):=\big\{z:\|z\|=r\big\}$, $d^c:=\dfrac{\sqrt{-1}}{4\pi}(\overline{\partial}-\partial)$, $\upsilon:=(dd^c\|z\|^2)^{m-1}$ and $\sigma:=d^c\log\|z\|^2\wedge(dd^c\log\|z\|^2)^{m-1}$.

Let F be a nonzero holomorphic function on $\mathbb{C}^m$. For an m-tuple $\alpha := (\alpha_1, \ldots, \alpha_m)$ of nonnegative integers, set $|\alpha| := \alpha_1 + \cdots + \alpha_m$ and $D^\alpha F := \dfrac{\partial^{|\alpha|} F}{\partial z_1^{\alpha_1} \ldots \partial z_m^{\alpha_m}}$. We define the map $v_F : \mathbb{C}^m \to \mathbb{N}_0$ by $v_F(z) := \max\{p : D^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < p\}$. Let k be a positive integer or $+\infty$. Define the map $v_F^{k)}$ of $\mathbb{C}^m$ into $\mathbb{N}_0$ by

$$v_F^{k)}(z) := \begin{cases} 0 & \text{if} \quad v_F(z) > k, \\ v_F(z) & \text{if} \quad v_F(z) \le k. \end{cases}$$

Let φ be a nonzero meromorphic function on $\mathbb{C}^m$. We define the map $v_\varphi^{k)}$ as follows: For each $z \in \mathbb{C}^m$, choose nonzero holomorphic functions F and G on a neighbourhood U of z such that $\varphi = \frac{F}{G}$ on U and $dim\big(F^{-1}(0) \cap G^{-1}(0)\big) \le m-2$. Then put $v_\varphi^{k)}(z) := v_F^{k)}(z)$. Set

$$\big|v_\varphi^{k)}\big| := \overline{\{z : v_\varphi^{k)}(z)0\}}.$$

Define

$$N^{k)}(r, v_\varphi) := \int_1^r \frac{n^{k)}(t)}{t^{2m-1}} dt, \quad (1 < r < +\infty)$$

where

$$n^{k)}(t) := \int\limits_{\left|v_\varphi^{k)}\right| \cap B(t)} v_\varphi^{k)} \upsilon \quad \text{for} \quad m \ge 2$$

and

$$n^{k)}(t) := \sum_{|z| \le t} v_\varphi^{k)}(z) \quad \text{for} \quad m = 1.$$

Set $N(r, v_\varphi) := N^{+\infty)}(r, v_\varphi)$. For l a positive integer or $+\infty$, set

$$N_l^{k)}(r, v_\varphi) := \int_1^r \frac{n_l^{k)}(t)}{t^{2m-1}} dt, \quad (1 < r < +\infty)$$

where $n_l^{k)}(t) := \int\limits_{\left|v_\varphi^{k)}\right| \cap B(t)} min\{v_\varphi^{k)}, l\} \upsilon$ for $m \ge 2$ and $n_l^{k)}(t) := \sum_{|z| \le t} min\{v_\varphi^{k)}(z), l\}$ for $m = 1$. Set $\overline{N}(r, v_\varphi) := N_1^{+\infty)}(r, v_\varphi)$ and

$\overline{N}^{k)}(r, v_\varphi) := N_1^{k)}(r, v_\varphi)$. For a closed subset A of a purely $(m-1)$-dimensional analytic subset of $\mathbb{C}^m$, we define

$$\bar{N}(r, A) := \int_1^r \frac{\bar{n}(t)}{t^{2m-1}} dt, \quad (1 < r < +\infty)$$

where

$$\bar{n}(t) := \begin{cases} \int\limits_{A \cap B(t)} \upsilon & \text{for } m \geq 2 \\ \#\left(A \cap B(t)\right) & \text{for } m = 1. \end{cases}$$

Let $f : \mathbb{C}^m \to \mathbb{C}P^n$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ on $\mathbb{C}P^n$, we take a reduced representation $f = (f_0 : \cdots : f_n)$ which means that each f_i is a holomorphic function on $\mathbb{C}^m$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ outside the analytic set $\{f_0 = \cdots = f_n = 0\}$ of codimension ≥ 2.

Set $\|f\| := (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$. The characteristic function of f is defined by

$$T_f(r) := \int\limits_{S(r)} \log \|f\| \sigma - \int\limits_{S(1)} \log \|f\| \sigma \ , \ r > 1.$$

For a nonzero meromorphic function φ on C^m, the characteristic function $T_\varphi(r)$ of φ is defined by considering φ as a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^1$.

Let $H = \{a_0 w_0 + \cdots + a_n w_n = 0\}$ be a hyperplane in $\mathbb{C}P^n$ such that $\mathrm{im} f \not\subseteq H$. Set $(f, H) := a_0 f_0 + \cdots + a_n f_n$. We define

$$N_f^{k)}(r, H) := N^{k)}(r, v_{(f,H)}) \quad \text{and} \quad N_{l,f}^{k)}(r, H) := N_l^{k)}(r, v_{(f,H)}).$$

Sometimes we write $\overline{N}_f^{k)}(r, H)$ for $N_{1,f}^{k)}(r, H)$, $N_{l,f}(r, H)$ for $N_{l,f}^{+\infty)}(r, H)$ and $N_f(r, H)$ for $N_{+\infty,f}^{+\infty)}(r, H)$.

Set $\psi_f(H) := \dfrac{\|f\|\left(|a_0|^2 + \cdots + |a_n|^2\right)^{1/2}}{(f, H)}$. We define the proximity function by

$$m_f(r, H) := \int\limits_{S(r)} \log |\, \psi_f(H) \,| \, \sigma - \int\limits_{S(1)} \log |\, \psi_f(H) \,| \, \sigma.$$

For a nonzero meromorphic function φ, the proximity function is defined by

$$m(r,\varphi) := \int\limits_{S(r)} \log^+ \mid \varphi \mid \sigma.$$

We note that $m(r,\varphi) = m_\varphi(r,+\infty) + O(1))$ ([4], p.135).

We state the First and the Second Main Theorem of Value Distribution Theory.

First Main Theorem. *Let $f : \mathbb{C}^m \to \mathbb{C}P^n$ be a meromorphic mapping and H a hyperplane in $\mathbb{C}P^n$ such that $\operatorname{im} f \not\subseteq H$. Then :*

$$N_f(r,H) + m_f(r,H) = T_f(r).$$

For a nonzero meromorphic function φ we have :

$$N(r, v_{\frac{1}{\varphi}}) + m(r,\varphi) = T_\varphi(r) + O(1).$$

Second Main Theorem. *Let $f : \mathbb{C}^m \to \mathbb{C}P^n$ be a linearly nondegenerate meromorphic mapping and $H_1, ..., H_q$ be hyperplanes in general position in $\mathbb{C}P^n$. Then:*

$$(q-n-1)T_f(r) \le \sum_{j=1}^{q} N_{n,f}(r,H_j) + o(T_f(r))$$

except for a set $E \subset (1,+\infty)$ of finite Lebesgue measure.

The following so-called logarithmic derivative lemma plays an essential role in Nevanlinna theory.

Theorem 2.1. ([5], Lemma 3.1) *Let φ be a non-constant meromorphic function on $\mathbb{C}^m$. Then for any i, $1 \le i \le m$, we have*

$$m\Big(r, \frac{\frac{\partial}{\partial z_i}\varphi}{\varphi}\Big) = o(T_\varphi(r)) \quad \text{as} \quad r \to \infty, \ \ r \notin E,$$

where $E \subset (1,+\infty)$ of finite Lebesgue measure.

Let F, G and H be nonzero meromorphic functions on $\mathbb{C}^m$. For each l, $1 \le l \le m$, we define the Cartan auxiliary function by

$$\Phi^l(F,G,H) := F \cdot G \cdot H \cdot \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{F} & \frac{1}{G} & \frac{1}{H} \\ \frac{\partial}{\partial z_l}\left(\frac{1}{F}\right) & \frac{\partial}{\partial z_l}\left(\frac{1}{G}\right) & \frac{\partial}{\partial z_l}\left(\frac{1}{H}\right) \end{vmatrix}.$$

By [4] (Proposition 3.4) we have the following

Theorem 2.2. *Let F, G, H be nonzero meromorphic functions on $\mathbb{C}^m$. Assume that $\Phi^l(F,G,H) \equiv 0$ and $\Phi^l\Big(\frac{1}{F}, \frac{1}{G}, \frac{1}{H}\Big) \equiv 0$ for all l, $1 \le l \le m$. Then one of the following assertions holds*

i) $F = G$ or $G = H$ or $H = F$.

ii) $\frac{F}{G}, \frac{G}{H}, \frac{H}{F}$ *are all constant.*

3 Proof of the Theorems

First of all, we need the following lemmas:

Lemma 1. *Let $f_1, ..., f_d$ be linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $\{H_j\}_{j=1}^q$ be hyperplanes in $\mathbb{C}P^n$. Then there exists a dense subset $\mathcal{C} \subset \mathbb{C}^{n+1} \setminus \{0\}$ such that for any $c = (c_0, ..., c_n) \in \mathcal{C}$, the hyperplane H_c defined by $c_0\omega_0 + ... + c_n\omega_n = 0$ satisfies*

$$dim(f_i^{-1}(H_j) \cap f_i^{-1}(H_c)) \leqslant m-2 \text{ for all } i \in \{1,...,d\} \text{ and } j \in \{1,...,q\}.$$

Proof. We refer to [5], Lemma 5.1. □

Let $f_1, f_2, f_3 \in F_k\big(\{H_j\}_{j=1}^q, f, 1\big)$, for $q \ge n+1$. Set

$$T(r) := T_{f_1}(r) + T_{f_2}(r) + T_{f_3}(r) .$$

For each $c \in \mathcal{C}$, set $F_{ic}^j := \dfrac{(f_i, H_j)}{(f_i, H_c)}$ for $i \in \{1,2,3\}$ and $j \in \{1,\dots,q\}$.

Lemma 2. *Assume that there exist $j_0 \in \{1,...,q\}, c \in \mathcal{C}, l \in \{1,...,m\}$ and a closed subset A of a purely $(m-1)$-dimensional analytic subset of $\mathbb{C}^m$*

satisfying
1) $\Phi^l_c := \Phi^l\big(F^{j_0}_{1c}, F^{j_0}_{2c}, F^{j_0}_{3c}\big) \not\equiv 0$, *and*
2) $min\big\{v^{k)}_{(f_1,H_{j_0})}, p\big\} = min\big\{v^{k)}_{(f_2,H_{j_0})}, p\big\} = min\big\{v^{k)}_{(f_3,H_{j_0})}, p\big\}$ *on* $\mathbb{C}^m \setminus A$,
where p *is a positive integer. Then*

$$\begin{aligned} 2\sum_{j=1,j\neq j_0}^{q} \overline{N}^{k)}_{f_i}(r,H_j) + N^{k)}_{p-1,f_i}(r,H_{j_0}) &\leq N(r,v_{\Phi^l_c}) + (p-1)\overline{N}(r,A) \\ &\leq \frac{k+2}{k+1}T(r) + (p+2)\overline{N}(r,A) + o(T(r)) \end{aligned}$$

for all $i \in \{1,2,3\}$.

Proof. Without loss of generality, we may assume that $l = 1$. For an arbitrary point $a \in \mathbb{C}^m \setminus A$ satisfying $v^{k)}_{(f_1,H_{j_0})}(a) > 0$, we have $v^{k)}_{(f_i,H_{j_0})}(a) > 0$ for all $i \in \{1,2,3\}$. We choose a such that $a \notin \bigcup_{i=1}^{3} f_i^{-1}(H_c)$. We distinguish between two cases, leading to equations (1) and (2).

Case 1. If $v_{(f_1,H_{j_0})}(a) \geq p$, then $v_{(f_i,H_{j_0})}(a) \geq p$, $i \in \{1,2,3\}$. This means that a is a zero point of $F^{j_0}_{ic}$ with multiplicity $\geq p$ for $i \in \{1,2,3\}$. We have

$$\begin{aligned} \Phi^1_c = {} & F^{j_0}_{1c}F^{j_0}_{3c}\frac{\partial}{\partial z_1}\Big(\frac{1}{F^{j_0}_{3c}}\Big) - F^{j_0}_{1c}F^{j_0}_{2c}\frac{\partial}{\partial z_1}\Big(\frac{1}{F^{j_0}_{2c}}\Big) \\ & + F^{j_0}_{2c}F^{j_0}_{1c}\frac{\partial}{\partial z_1}\Big(\frac{1}{F^{j_0}_{1c}}\Big) - F^{j_0}_{2c}F^{j_0}_{3c}\frac{\partial}{\partial z_1}\Big(\frac{1}{F^{j_0}_{3c}}\Big) \\ & + F^{j_0}_{3c}F^{j_0}_{2c}\frac{\partial}{\partial z_1}\Big(\frac{1}{F^{j_0}_{2c}}\Big) - F^{j_0}_{3c}F^{j_0}_{1c}\frac{\partial}{\partial z_1}\Big(\frac{1}{F^{j_0}_{1c}}\Big). \end{aligned}$$

On the other hand $F^{j_0}_{1c}F^{j_0}_{3c}\frac{\partial}{\partial z_1}\Big(\frac{1}{F^{j_0}_{3c}}\Big) = \frac{-F^{j_0}_{1c}\frac{\partial}{\partial z_1}F^{j_0}_{3c}}{F^{j_0}_{3c}}$, so a is a zero point of $F^{j_0}_{1c}F^{j_0}_{3c}\frac{\partial}{\partial z_1}\Big(\frac{1}{F^{j_0}_{3c}}\Big)$ with multiplicity $\geq p-1$. By applying the same argument also to all other combinations of indices, we see that
a is a zero point of Φ^1_c with multiplicity $\geq p-1$. (1)

Case 2. If $v_{(f_1,H_{j_0})}(a) \leq p$, then $p_0 := v_{(f_1,H_{j_0})}(a) = v_{(f_2,H_{j_0})}(a) = v_{(f_3,H_{j_0})}(a) \leq p$. There exists a neighborhood U of a such that $v_{(f_1,H_{j_0})} \leq p$ on U. Indeed, otherwise there exist a sequence $\{a_s\}_{s=1}^{\infty} \subset \mathbb{C}^m$, $\lim\limits_{s\to\infty} a_s = a$ and

$v_{(f_1,H_{j_0})}(a_s) \geq p+1$ for all s. By the definition, we have $D^\beta(f_1,H_{j_0})(a_s)=0$ for all $|\beta| < p+1$. So $D^\beta(f_1,H_{j_0})(a) = \lim_{s\to\infty} D^\beta(f_1,H_{j_0})(a_s) = 0$ for all $|\beta| < p+1$. Thus $v_{(f_1,H_{j_0})}(a) \geq p+1$. This is a contradiction. Hence $v_{(f_1,H_{j_0})} \leq p$ on U.

We can choose U such that $U \cap A = \emptyset$, $v_{(f_i,H_{j_0})} \leq p$ on U and (f_i,H_c) has no zero point on U for all $i \in \{1,2,3\}$. Then $v_{F_{1c}^{j_0}} = v_{F_{2c}^{j_0}} = v_{F_{3c}^{j_0}} \leq p$ on U. So $U \cap \{F_{1c}^{j_0}=0\} = U \cap \{F_{2c}^{j_0}=0\} = U \cap \{F_{3c}^{j_0}=0\}$. Choose a such that a is regular point of $U \cap \{F_{1c}^{j_0}=0\}$. By shrinking U we may assume that there exists a holomorphic function h on U such that dh has no zero point and $F_{ic}^{j_0} = h^{p_0}u_i$ on U, where $u_i (i=1,2,3)$ are nowhere vanishing holomorphic functions on U (note that $v_{F_{1c}^{j_0}}(a) = v_{F_{2c}^{j_0}}(a) = v_{F_{3c}^{j_0}}(a) = p_0$). We have

$$\Phi_c^1 = u_1 \frac{\big(u_3\frac{\partial}{\partial z_1}u_2 - u_2\frac{\partial}{\partial z_1}u_3\big)h^{p_0}}{u_2u_3} + u_2 \frac{\big(u_1\frac{\partial}{\partial z_1}u_3 - u_3\frac{\partial}{\partial z_1}u_1\big)h^{p_0}}{u_3u_1} + u_3 \frac{\big(u_2\frac{\partial}{\partial z_1}u_1 - u_1\frac{\partial}{\partial z_1}u_2\big)h^{p_0}}{u_1u_2}.$$

So, we have
a is a zero point of Φ_c^1 with mulitplicity $\geq p_0$ (2)

By (1), (2) and our choice of a, there exists an analytic set $M \subset \mathbb{C}^m$ with codimension ≥ 2 such that $v_{\Phi_c^1} \geq \min\{v_{(f_1,H_{j_0})}\ , p-1\}$ on

$$\big\{z : v^{k)}_{(f_1,H_{j_0})}(z) > 0\big\} \setminus (M \cup A). \tag{3}$$

For each $j \in \{1,\ldots,q\} \setminus \{j_0\}$, let a (depending on j) be an arbitrary point in $\mathbb{C}^m$ such that $v^{k)}_{(f_1,H_j)}(a) > 0$ (if there exist any). Then $v^{k)}_{(f_i,H_j)}(a) > 0$ for all $i \in \{1,2,3\}$, since $f_1,f_2,f_3 \in F_k\big(\{H_j\}_{j=1}^q, f, 1\big)$. We can choose $a \notin f_i^{-1}(H_c) \cup f_i^{-1}(H_{j_0})$, $i=1,2,3$. Then there exists a neighborhood U of a such that $v_{(f_i,H_j)} \leq k$ on U and (f_i,H_{j_0}), (f_i,H_c) ($i=1,2,3$) have no zero point on U. We have $B := f_1^{-1}(H_j) \cap U = f_2^{-1}(H_j) \cap U = f_3^{-1}(H_j) \cap U$ and $\dfrac{1}{F_{1c}^{j_0}} = \dfrac{1}{F_{2c}^{j_0}} = \dfrac{1}{F_{3c}^{j_0}}$ on B. Choose a such that a is a regular point of B. By shrinking U, we may assume that there exists a holomorphic function h on U such that dh has no zero point and $U \cap \{h=0\} = B$. Then $\dfrac{1}{F_{2c}^{j_0}} - \dfrac{1}{F_{1c}^{j_0}} = h\varphi_2$ and $\dfrac{1}{F_{3c}^{j_0}} - \dfrac{1}{F_{1c}^{j_0}} = h\varphi_3$ on U where φ_2, φ_3 are holomorphic functions on U.

Hence, we get

$$\begin{aligned}\Phi_c^1 &= F_{1c}^{j_0}F_{2c}^{j_0}F_{3c}^{j_0}\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{F_{1c}^{j_0}} & h\varphi_2 & h\varphi_3 \\ \frac{\partial}{\partial z_1}\left(\frac{1}{F_{1c}^{j_0}}\right) & \varphi_2\frac{\partial}{\partial z_1}h + h\frac{\partial}{\partial z_1}\varphi_2 & \varphi_3\frac{\partial}{\partial z_1}h + h\frac{\partial}{\partial z_1}\varphi_3 \end{vmatrix} \\ &= F_{1c}^{j_0}F_{2c}^{j_0}F_{3c}^{j_0}h^2\begin{vmatrix} \varphi_2 & \varphi_3 \\ \frac{\partial}{\partial z_1}\varphi_2 & \frac{\partial}{\partial z_1}\varphi_3 \end{vmatrix}.\end{aligned}$$

Hence, a is a zero point of Φ_c^1 with multiplicity ≥ 2. Thus, for each $j \in \{1, \ldots, q\} \setminus \{j_0\}$, there exists an analytic set $N \subset \mathbb{C}^m$ with codimension ≥ 2 such that $v_{\Phi_c^1} \geq 2$ on

$$\big\{z : v^{k)}_{(f_1,H_j)}(z) > 0\big\} \setminus N. \tag{4}$$

By (3) and (4), we have

$$2\sum_{j=1,j\neq j_0}^{q} \overline{N}^{k)}_{f_1}(r,H_j) + N^{k)}_{p-1,f_1}(r,H_{j_0}) \leq N(r,v_{\Phi_c^1}) + (p-1)\overline{N}(r,A).$$

Similarly, we have

$$2\sum_{j=1,j\neq j_0}^{q} \overline{N}^{k)}_{f_i}(r,H_j) + N^{k)}_{p-1,f_i}(r,H_{j_0}) \leq N(r,v_{\Phi_c^1}) + (p-1)\overline{N}(r,A), \quad i = 1,2,3. \tag{5}$$

Let a be an arbitrary zero point of some $F_{ic}^{j_0}$, $a \notin f_i^{-1}(H_c)$, say $i = 1$. We have

$$\begin{aligned}\Phi_c^1 &= \big(F_{2c}^{j_0} - F_{3c}^{j_0}\big)F_{1c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{1c}^{j_0}}\Big) + \big(F_{3c}^{j_0} - F_{1c}^{j_0}\big)F_{2c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{2c}^{j_0}}\Big) \\ &\quad + \big(F_{1c}^{j_0} - F_{2c}^{j_0}\big)F_{3c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{3c}^{j_0}}\Big).\end{aligned} \tag{6}$$

So we have

$$v_{\frac{1}{\Phi_c^1}}(a) \leq 1 + \max\{v_{\frac{1}{F_{ic}^{j_0}}}(a), i = 2,3\} \leq 1 + v_{\frac{1}{F_{2c}^{j_0}}}(a) + v_{\frac{1}{F_{3c}^{j_0}}}(a) .$$

Furthermore, if $0 < v_{F_{1c}^{j_0}}(a) \leq k$ $\big($and, hence, $v^{k)}_{(f_1,H_{j_0})}(a) > 0\big)$ and $a \notin A$, then by (3) we may assume that $v_{\frac{1}{\Phi_c^1}}(a) = 0$ (outside an analytic set of

codimension ≥ 2). (7)
Let a be an arbitrary pole of all $F_{ic}^{j_0}$, $i = 1, 2, 3$. By (6) we have

$$v_{\frac{1}{\Phi_c^1}}(a) \leq \max\{v_{\frac{1}{F_{ic}^{j0}}}(a), i = 1, 2, 3\} + 1 < \sum_{i=1}^{3} v_{\frac{1}{F_{ic}^{j0}}}(a) \tag{8}$$

It follows from (6) that a pole of Φ_c^1 is a zero or a pole of some $F_{ic}^{j_0}$. Thus, by (6), (7) and (8), we have

$$\begin{aligned} N\Big(r, v_{\frac{1}{\Phi_c^1}}\Big) &\leq \sum_{i=1}^{3} N\Big(r, v_{\frac{1}{F_{ic}^{j0}}}\Big) + \sum_{i=1}^{3} \Big(\overline{N}\big(r, v_{F_{ic}^{j_0}}\big) - \overline{N}^{k)}\big(r, v_{F_{ic}^{j_0}}\big)\Big) + 3\overline{N}(r, A) \\ &\leq \sum_{i=1}^{3} N\Big(r, v_{\frac{1}{F_{ic}^{j0}}}\Big) + \frac{1}{k+1} \sum_{i=1}^{3} N\big(r, v_{F_{ic}^{j_0}}\big) + 3\overline{N}(r, A) \\ &\leq \sum_{i=1}^{3} N\big(r, v_{\frac{1}{F_{ic}^{j0}}}\big) + \frac{1}{k+1} \sum_{i=1}^{3} T_{F_{ic}^{j_0}}(r) + 3\overline{N}(r, A) \\ &\leq \sum_{i=1}^{3} N\big(r, v_{\frac{1}{F_{ic}^{j0}}}\big) + \frac{1}{k+1} T(r) + 3\overline{N}(r, A) + O(1). \end{aligned} \tag{9}$$

We have

$$\begin{aligned} \Phi_c^1 = F_{1c}^{j_0}\Big[F_{3c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{3c}^{j_0}}\Big) - F_{2c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{2c}^{j_0}}\Big)\Big] \\ + F_{2c}^{j_0}\Big[F_{1c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{1c}^{j_0}}\Big) - F_{3c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{3c}^{j_0}}\Big)\Big] \\ + F_{3c}^{j_0}\Big[F_{2c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{2c}^{j_0}}\Big) - F_{1c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{1c}^{j_0}}\Big)\Big] \end{aligned}$$

so $m(r, \Phi_c^1) \leq \sum_{i=1}^{3} m(r, F_{ic}^{j_0}) + 2\sum_{i=1}^{3} m\Big(r, F_{ic}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{ic}^{j0}}\Big)\Big) + 0(1)$. By Theorem 2.1, we have

$$m\Big(r, F_{ic}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{ic}^{j_0}}\Big)\Big) = o\big(T_{F_{ic}^{j_0}}(r)\big)$$

Thus, we get

$$m(r, \Phi_c^1) \leq \sum_{i=1}^{3} m(r, F_{ic}^{j_0}) + o(T(r)), \tag{10}$$

(note that $T_{F^{j_0}_{ic}}(r) \le T_{f_i}(r) + O(1)$).
By (9) , (10) and by the First Main Theorem, we have

$$\begin{aligned}
N(r, v_{\Phi^1_c}) &\le T_{\Phi^1_c}(r) + O(1) = N\big(r, v_{\frac{1}{\Phi^1_c}}\big) + m(r, \Phi^1_c) + O(1) \\
&\le \sum_{i=1}^{3} \Big(N\big(r, v_{\frac{1}{F^{j_0}_{ic}}}\big) + m(r, F^{j_0}_{ic}) \Big) + \frac{1}{k+1} T(r) + 3\bar{\bar{N}}(r, A) + o(T(r)) \\
&\le \sum_{i=1}^{3} T_{F^{j_0}_{ic}}(r) + \frac{1}{k+1} T(r) + 3\bar{\bar{N}}(r, A) + o(T(r)) \\
&\le \sum_{i=1}^{3} T_{f_i}(r) + \frac{1}{k+1} T(r) + 3\bar{\bar{N}}(r, A) + o(T(r)) \\
&= \frac{k+2}{k+1} T(r) + 3\bar{\bar{N}}(r, A) + o(T(r)). \qquad (11)
\end{aligned}$$

By (5) and (11) we get Lemma 2. □

The following lemma is a variant of the Second Main Theorem without taking account of multiplicities of order $> k$ in the counting functions.

Lemma 3. *Let f be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $\{H_j\}_{j=1}^q (q \ge n+2)$ be hyperplanes in $\mathbb{C}P^n$ in general position. Take a positive integer k with $\frac{qn}{q-n-1} \le k \le +\infty$. Then*

$$\begin{aligned}
T_f(r) &\le \frac{k}{(q-n-1)(k+1) - qn} \sum_{j=1}^{q} N^{k)}_{n,f}(r, H_j) + o(T_f(r)) \\
&\le \frac{nk}{(q-n-1)(k+1) - qn} \sum_{j=1}^{q} \overline{N}^{k)}_f(r, H_j) + o(T_f(r))
\end{aligned}$$

for all $r > 1$ except a set E of finite Lebesgue measure.

Proof. By the First and the Second Main Theorems, we have

$$(q-n-1) T_f(r) \le \sum_{j=1}^{q} N_{n,f}(r, H_j) + o\big(T_f(r)\big)$$

$$\leq \frac{k}{k+1}\sum_{j=1}^{q} N^{k)}_{n,f}(r,H_j) + \frac{n}{k+1}\sum_{j=1}^{q} N_f(r,H_j) + o\big(T_f(r)\big)$$

$$\leq \frac{k}{k+1}\sum_{j=1}^{q} N^{k)}_{n,f}(r,H_j) + \frac{qn}{k+1}T_f(r) + o\big(T_f(r)\big), \quad r \notin E,$$

which impies that

$$\Big(q-n-1-\frac{qn}{k+1}\Big)T_f(r) \leq \frac{k}{k+1}\sum_{j=1}^{q} N^{k)}_{n,f}(r,H_j) + o\big(T_f(r)\big) \ .$$

Thus, we have

$$\begin{aligned} T_f(r) \ &\leq \ \frac{k}{(q-n-1)(k+1)-qn}\sum_{j=1}^{q} N^{k)}_{n,f}(r,H_j) + o(T_f(r)) \\ &\leq \ \frac{nk}{(q-n-1)(k+1)-qn}\sum_{j=1}^{q} \overline{N}^{k)}_{f}(r,H_j) + o(T_f(r)) \qquad \square \end{aligned}$$

Proof of Theorem 1. Assume that there exist three distinct mappings $f_1, f_2, f_3 \in F_k(\{H_j\}_{j=1}^q, f, p)$. Denote by Q the set which contains all indices $j \in \{1,...,q\}$ satisfying $\Phi^l\big(F^j_{1c}, F^j_{2c}, F^j_{3c}\big) \not\equiv 0$ for some $c \in \mathcal{C}$ and some $l \in \{1,...,m\}$. We now prove that

$$\#(\{1,...,q\}\backslash Q) \geq 3n-1. \tag{12}$$

For the proof of (12) we distinguish three cases:

Case 1. $1 \leq n \leq 3, q = 3n+1, p = 2, k \geq 23n$.
Suppose that (12) does not hold, then $\#Q \geq 3$. For each $j_0 \in Q$, by Lemma 2 (with $A = \emptyset$, $p = 2$) we have

$$2\sum_{j=1, j\neq i_0}^{q} \overline{N}^{k)}_{f_i}(r,H_j) + \overline{N}^{k)}_{f_i}(r,H_{j_0}) \leq \frac{k+2}{k+1}T(r) + o(T(r)) \ , \ i = 1,2,3. \tag{13}$$

By (13) and Lemma 3 we have

$$\Big(q-n-1-\frac{qn}{k+1}\Big)T_{f_i}(r) \le \frac{nk}{k+1}\sum_{j=1}^{q}\overline{N}^{k)}_{f_i}(r,H_j)+o\big(T_{f_i}(r)\big)$$
$$\le \ \frac{nk(k+2)}{2(k+1)^2}T(r)+\frac{nk}{2(k+1)}\overline{N}^{k)}_{f_i}(r,H_{j_0})+o\big(T(r)\big), i=1,2,3.$$

Thus, we obtain

$$\Big(q-n-1-\frac{qn}{k+1}\Big)T(r) \le \frac{3nk(k+2)}{2(k+1)^2}T(r)+\frac{nk}{2(k+1)}\sum_{i=1}^{3}\overline{N}^{k)}_{f_i}(r,H_{j_0})+o\big(T(r)\big),$$

which implies

$$\big[2(q-n-1)(k+1)^2-2qn(k+1)-3nk(k+2)\big]T(r)$$
$$\le nk(k+1)\sum_{i=1}^{3}\overline{N}^{k)}_{f_i}(r,H_{j_0})+o(T(r))=3nk(k+1)\overline{N}^{k)}_{f_i}(r,H_{j_0})+o(T(r)) \ .$$

Hence, we have

$$\liminf_{r\to\infty\ r\notin E}\frac{\overline{N}^{k)}_{f_i}(r,H_{j_0})}{T(r)} \ge \frac{2(q-n-1)(k+1)^2-2qn(k+1)-3nk(k+2)}{3nk(k+1)}$$
$$=\frac{k^2-6nk-6n+2}{3k(k+1)} \ , i\in\{1,2,3\}. \tag{14}$$

Set

$$A_i:=\big\{r>1:T_{f_i}(r)=min\big\{T_{f_1}(r),T_{f_2}(r),T_{f_3}(r)\big\}\big\},\quad i\in\{1,2,3\}.$$

Then $A_1\cup A_2\cup A_3=(1,+\infty)$. Without loss of generality, we may assume that the Lebesgue measure of A_1 is infinite. By (14) we have

$$\liminf_{r\to\infty\ r\in A_1\setminus E}\frac{\overline{N}^{k)}_{f_1}(r,H_{j_0})}{T_{f_1}(r)} \ge \frac{k^2-6nk-6n+2}{k(k+1)} \ , j_0\in Q.$$

Take three distinct indices $j_1,j_2,j_3\in Q$ (note that $\#Q\ge 3$). Then we have

$$\liminf_{r\to\infty\ r\in A_1\setminus E}\frac{\overline{N}^{k)}_{f_1}(r,H_{j_1})+\overline{N}^{k)}_{f_1}(r,H_{j_2})+\overline{N}^{k)}_{f_1}(r,H_{j_3})}{T_{f_1}(r)} \ge \frac{3(k^2-6nk-6n+2)}{k(k+1)},$$

which implies that

$$\liminf_{r\to\infty\ r\in A_1\setminus E}\frac{\sum\limits_{j=1}^{q}\overline{N}^{k)}_{f_1}(r,H_j)}{T_{f_1}(r)}\geq\frac{3(k^2-6nk-6n+2)}{k(k+1)}\ . \tag{15}$$

Since $f_1\not\equiv f_2$ there exists $c\in\mathcal{C}$ such that $\frac{(f_1,H_1)}{(f_1,H_c)}\not\equiv\frac{(f_2,H_1)}{(f_2,H_c)}$. Indeed, otherwise by Lemma 1 we have that $\frac{(f_1,H_1)}{(f_1,H)}\equiv\frac{(f_2,H_1)}{(f_2,H)}$ for all hyperplanes H in $\mathbb{C}P^n$. In particular $\frac{(f_1,H_1)}{(f_1,H_j)}\equiv\frac{(f_2,H_1)}{(f_2,H_j)}$ for all $j=2,...,n+1$. We choose homogeneous coordinates $(\omega_0:\cdots:\omega_n)$ on $\mathbb{C}P^n$ with $H_j=\{\omega_j=0\}$ $(1\leq j\leq n+1)$ and take reduced representations: $f_1=(f_{1_1}:\cdots:f_{1_{n+1}})$, $f_2=(f_{2_1}:\cdots:f_{2_{n+1}})$. Then

$$\begin{cases}\dfrac{f_{1_j}}{f_{1_1}}=\dfrac{f_{2_j}}{f_{2_1}}\\ (j=2,\ldots,n+1)\end{cases}\quad\Rightarrow\quad\frac{f_{1_1}}{f_{2_1}}=\cdots=\frac{f_{1_{n+1}}}{f_{2_{n+1}}}\ \Rightarrow f_1\equiv f_2.$$

This is a contradiction.

Since $dim\ (f_i^{-1}(H_1)\cap f_i^{-1}(H_c))\leq m-2$ we have

$$\begin{aligned}T_{\frac{(f_i,H_1)}{(f_i,H_c)}}(r)&=\int\limits_{S(r)}\log\left(|(f_i,H_1)|^2+|(f_i,H_c)|^2\right)^{\frac{1}{2}}\sigma\quad+O(1)\\&\leq\int\limits_{S(r)}\log\|f_i\|\sigma+O(1)=T_{f_i}(r)+O(1)\ ,\ i=1,2,3.\end{aligned}$$

Since $f_1=f_2$ on $\bigcup\limits_{j=1}^{q}\left\{z:v^{k)}_{(f_1,H_j)}(z)>0\right\}$ and $dim\left\{z:v^{k)}_{(f_1,H_i)}(z)>0\ \ and\ \ v^{k)}_{(f_1,H_j)}(z)>0\right\}\leq m-2$ for all $i\neq j$, ? ?we have

$$\begin{aligned}&\sum_{j=1}^{q}\overline{N}^{k)}_{f_1}(r,H_j)\leq N\Big(r,v_{\frac{(f_1,H_1)}{(f_1,H_c)}-\frac{(f_2,H_1)}{(f_2,H_c)}}\Big)\leq T_{\frac{(f_1,H_1)}{(f_1,H_c)}-\frac{(f_2,H_1)}{(f_2,H_c)}}(r)\ \ +0(1)\\&\leq T_{\frac{(f_1,H_1)}{(f_1,H_c)}}(r)+T_{\frac{(f_2,H_1)}{(f_2,H_c)}}(r)+0(1)\leq T_{f_1}(r)+T_{f_2}(r)+0(1),\end{aligned}$$

which implies

$$\liminf_{r\to\infty}\frac{T_{f_1}(r)+T_{f_2}(r)}{\sum\limits_{j=1}^{q}\overline{N}^{k)}_{f_1}(r,H_j)}\geq 1.$$

On the other hand, by Lemma 3, we have

$$\Big(q-n-1-\frac{qn}{k+1}\Big)T_{f_i}(r) \le \frac{nk}{k+1}\sum_{j=1}^{q}\overline{N}^{k)}_{f_i}(r,H_j)+o\big(T_{f_i}(r)\big)$$
$$= \frac{nk}{k+1}\sum_{j=1}^{q}\overline{N}^{k)}_{f_1}(r,H_j)+o\big(T_{f_i}(r)\big)\ ,$$

which implies

$$\limsup_{r\to\infty\ r\notin E}\frac{T_{f_i}(r)}{\sum_{j=1}^{q}\overline{N}^{k)}_{f_1}(r,H_j)} \le \frac{nk}{(q-n-1)(k+1)-qn}\ ,\ i=1,2,3\ .$$

Hence, we obtain

$$\limsup_{r\to\infty\ r\in A_1\setminus E}\frac{T_{f_1}(r)}{\sum_{j=1}^{q}\overline{N}^{k)}_{f_1}(r,H_j)} = \limsup_{r\to\infty\ r\in A_1\setminus E}\Big(\frac{T_{f_1}(r)+T_{f_2}(r)}{\sum_{j=1}^{q}\overline{N}^{k)}_{f_1}(r,H_j)}-\frac{T_{f_2}(r)}{\sum_{j=1}^{q}\overline{N}^{k)}_{f_1}(r,H_j)}\Big)$$

$$\ge \liminf_{r\to\infty\ r\in A_1\setminus E}\frac{T_{f_1}(r)+T_{f_2}(r)}{\sum_{j=1}^{q}\overline{N}^{k)}_{f_1}(r,H_j)} - \limsup_{r\to\infty\ r\in A_1\setminus E}\frac{T_{f_2}(r)}{\sum_{j=1}^{q}\overline{N}^{k)}_{f_1}(r,H_j)} \ge 1-\frac{nk}{(q-n-1)(k+1)-qn}$$

Consequently, we get

$$\liminf_{r\to\infty\ r\in A_1\setminus E}\frac{\sum_{j=1}^{q}\overline{N}^{k)}_{f_1}(r,H_j)}{T_{f_1}(r)} \le \frac{(q-n-1)(k+1)-qn}{(q-n-1)(k+1)-qn-nk} = \frac{2k+1-3n}{k+1-3n} \qquad (16)$$

By (15) and (16) we have

$$\frac{3(k^2-6nk-6n+2)}{k(k+1)} \le \frac{2k+1-3n}{k+1-3n}.$$

This contradicts $k\ge 23n$. Thus, we get (12) in this case.

Case 2. $4 \le n \le 6$, $q = 3n$, $p = 2$, $k \ge \frac{(6n-1)n}{n-3}$.
Suppose that (12) does not hold, then there exists $j_0 \in Q$. By Lemma 2 (with $A = \emptyset, p = 2$) we have

$$2 \sum_{j=1, j\neq j_0}^{3n} \overline{N}_{f_i}^{k)}(r, H_j) + \overline{N}_{f_i}^{k)}(r, H_{j_0}) \le \frac{k+2}{k+1} T(r) + o(T(r)) \ , \ i = 1, 2, 3 \ .$$

On the other hand, by Lemma 3 we have

$$\begin{aligned} \sum_{j=1, j\neq j_0}^{3n} \overline{N}_{f_i}^{k)}(r, H_j) + o(T_{f_i}(r)) &\ge \frac{(2n-2)(k+1) - (3n-1)n}{nk} T_{f_i}(r), \text{ and} \\ \sum_{j=1}^{3n} \overline{N}_{f_i}^{k)}(r, H_j) + o(T_{f_i}(r)) &\ge \frac{(2n-1)(k+1) - 3n^2}{nk} T_{f_i}(r), \end{aligned}$$

which implies that

$$2 \sum_{j=1, j\neq j_0}^{3n} \overline{N}_{f_i}^{k)}(r, H_j) + \overline{N}_{f_i}^{k)}(r, H_{j_0}) + o(T_{f_i}(r)) \ge \frac{(4n-3)(k+1) - (6n-1)n}{nk} T_{f_i}(r)$$

Hence, we have

$$\frac{(4n-3)(k+1) - (6n-1)n}{nk} T_{f_i}(r) \le \frac{k+2}{k+1} T(r) + o(T(r)) \ , \ i = 1, 2, 3.$$

Consequently, we get

$$\frac{(4n-3)(k+1) - (6n-1)n}{nk} T(r) \le \frac{3(k+2)}{k+1} T(r) + o(T(r)),$$

which implies that

$$\begin{aligned} ((4n-3)(k+1) - (6n-1)n)\, T(r) &\le \frac{3nk(k+2)}{k+1} T(r) + o(T(r)) \\ &\le 3n(k+1) T(r) + o(T(r). \end{aligned}$$

Hence, we obtain $k+1 \le \frac{(6n-1)n}{n-3}$. This is a contradiction. Thus, we get (12) in this case.

Case 3. $n \geq 7$, $q = 3n-1$, $p = 1$, $k \geq \frac{(6n-4)n}{n-6}$.
Suppose that (12) does not hold, then there exists $j_0 \in Q$. By Lemma 2 (with $A = \emptyset, p = 1$) we have

$$2 \sum_{j=1, j \neq j_0}^{3n-1} \overline{N}_{f_i}^{k)}(r, H_j) \leq \frac{k+2}{k+1} T(r) + o(T(r)) \ , \ i = 1, 2, 3$$

(note that $N_{0,f_i}^{k)}(r, H_{j_0}) = 0$). On the other hand, by Lemma 3, we have

$$2 \sum_{j=1, j \neq j_0}^{3n-1} \overline{N}_{f_i}^{k)}(r, H_j) + o(T_{f_i}(r)) \geq 2 \frac{(2n-3)(k+1) - (3n-2)n}{nk} T_{f_i}(r)$$

Hence, we get

$$\frac{2[(2n-3)(k+1) - (3n-2)n)]}{nk} T_{f_i}(r) \leq \frac{k+2}{k+1} T(r) + o(T(r)) \ ,$$

which implies

$$((4n-6)(k+1) - (6n-4)n) T_{f_i}(r) \leq \frac{nk(k+2)}{k+1} T(r) + o(T(r)), \ i = 1, 2, 3.$$

Hence, we have

$$\begin{aligned} ((4n-6)(k+1) - (6n-4)n) T(r) \leq & \quad \frac{3nk(k+2)}{k+1} T(r) + o(T(r)) \\ \leq & \quad 3n(k+1) T(r) + o(T(r)). \end{aligned}$$

Thus, we obtain

$$(4n-6)(k+1) - (6n-4)n \leq 3n(k+1)$$

implying

$$k+1 \leq \frac{(6n-4)n}{n-6},$$

which is a contradiction. Thus, we get (12) in this case.

So, for any case we have $\#(\{1, \ldots, q\} \setminus Q) \geq 3n-1$. Without loss of generality, we may assume that $1, \ldots, 3n-1 \notin Q$. We have
$\Phi^l(F_{1c}^j, F_{2c}^j, F_{3c}^j) \equiv 0$ for all $c \in \mathcal{C}$, $l \in \{1, ..., m\}$, $j \in \{1, \ldots, 3n-1\}$.

On the other hand, $\mathcal{C}$ is dense in $\mathbb{C}^{n+1}$. Hence, $\Phi^l\big(F^j_{1c}, F^j_{2c}, F^j_{3c}\big) \equiv 0$ for all $c \in \mathbb{C}^{n+1}\setminus\{0\}$, $l \in \{1, ..., m\}$, $j \in \{1, \dots, 3n-1\}$. In particular (for $H_c = H_i$) we have

$$\Phi^l\left(\frac{(f_1,H_j)}{(f_1,H_i)},\ \frac{(f_2,H_j)}{(f_2,H_i)},\ \frac{(f_3,H_j)}{(f_3,H_i)}\right) \equiv 0$$

for all $1 \le i \ne j \le 3n-1, \quad l \in \{1, \dots, m\}$. (17)

In the following we distinguish between the cases $n = 1$ and $n \geq 2$.

Case 1. If $n = 1$, then $a_j := H_j (j = 1, 2, 3, 4)$ are distinct points in $\mathbb{C}P^1$. We have that

$$g_1 := \frac{(f_1,a_1)}{(f_1,a_2)},\ \ g_2 := \frac{(f_2,a_1)}{(f_2,a_2)},\ \ g_3 := \frac{(f_3,a_1)}{(f_3,a_2)}$$

are distinct nonconstant meromorphic functions. By (17) and by Theorem 2.2, there exist constants α, β such that

$$g_2 = \alpha g_1 \quad , \ g_3 = \beta g_1 \ , (\alpha, \beta \notin \{1, \infty, 0\}, \alpha \neq \beta) \tag{18}$$

We have $v_{(f_1,a_3)} \geq k+1$ on $\{z : (f_1, a_3)(z) = 0\}$: Indeed, otherwise there exists z_0 such that $0 < v_{(f_1,a_3)}(z_0) \le k$. Then $v^{k)}_{(f_i,a_3)}(z_0) > 0$, for all $i \in \{1, 2, 3\}$. We have $(f_1, a_3)(z_0) = (f_2, a_3)(z_0) = 0 \Rightarrow \ f_1(z_0) = f_2(z_0) = a^*_3$, where we denote $a^*_j := (a_{j_1} : -a_{j_0})$ for every point $a_j = (a_{j_0} : a_{j_1}) \in \mathbb{C}P^1$. So $g_1(z_0) = g_2(z_0) = \dfrac{(a^*_3, a_1)}{(a^*_3, a_2)} \neq 0, \infty$ (note that $a_3 \neq a_1$, $a_3 \neq a_2$). So, by (18) we have $\alpha = 1$. This is a contradiction. Thus $v_{(f_1,a_3)} \geq k + 1$ on $\{z : (f_1, a_3)(z) = 0\}$. Similarly, $v_{(f_i,a_j)} \geq k + 1$ on $\{z : (f_i, a_j)(z) = 0\}$ for $i \in \{1, 2, 3\}$, $j \in \{3, 4\}$.

Set $b_1 = \alpha\dfrac{(a^*_3, a_1)}{(a^*_3, a_2)}$, $b_2 = \dfrac{\alpha}{\beta}\dfrac{(a^*_3, a_1)}{(a^*_3, a_2)}$, $b_3 = \dfrac{(a^*_3, a_1)}{(a^*_3, a_2)}$. Then we have

$$v_{g_2-b_3} = v_{\frac{(f_2,a_3)(a^*_1,a_2)}{(f_2,a_2)(a^*_3,a_2)}} \geq k+1 \ \text{ on } \ \{z : (g_2 - b_3)(z) = 0\},$$

$$v_{g_2-b_1} = v_{g_1-\frac{1}{\alpha}b_1} = v_{\frac{(f_1,a_3)(a^*_1,a_2)}{(f_1,a_2)(a^*_3,a_2)}} \geq k+1 \ \text{ on } \ \{z : (g_2 - b_1)(z) = 0\}, \text{and}$$

$$v_{g_2-b_2} = v_{g_3-\frac{\beta}{\alpha}b_2} = v_{\frac{(f_3,a_3)(a^*_1,a_2)}{(f_3,a_2)(a^*_3,a_2)}} \geq k+1 \ \text{ on } \ \{z : (g_2 - b_2)(z) = 0\}.$$

Since the points b_1, b_2, b_3 are distinct, by the First and the Second Main Theorem, we have

$$\begin{aligned}T_{g_2}(r) &\le \sum_{j=1}^{3} \overline{N}\big(r, v_{g_2-b_j}\big) + o\big(T_{g_2}(r)\big)\\ &\le \frac{1}{k+1}\sum_{j=1}^{3} N\big(r, v_{g_2-b_j}\big) + o\big(T_{g_2}(r)\big)\\ &\le \frac{3}{k+1}T_{g_2}(r) + o\big(T_{g_2}(r)\big)\ .\end{aligned}$$

This contradicts $k \ge 23$.

Case 2. If $n \ge 2$, for each $1 \le i \ne j \le 3n-1$, by (17) and Theorem 2.2., there exists a constant α_{ij} such that

$$\frac{(f_2, H_j)}{(f_2, H_i)} = \alpha_{ij}\frac{(f_1, H_j)}{(f_1, H_i)} \text{ or } \frac{(f_3, H_j)}{(f_3, H_i)} = \alpha_{ij}\frac{(f_1, H_j)}{(f_1, H_i)} \text{ or } \frac{(f_3, H_j)}{(f_3, H_i)} = \alpha_{ij}\frac{(f_2, H_j)}{(f_2, H_i)} \tag{19}$$

We now prove that $\alpha_{ij} = 1$ for all $1 \le i \ne j \le 3n-1$. Indeed, if there exists $\alpha_{i_0j_0} \ne 1$, without loss of generality, we may assume that $\frac{(f_2, H_{j_0})}{(f_2, H_{i_0})} = \alpha_{i_0j_0}\frac{(f_1, H_{j_0})}{(f_1, H_{i_0})}$. On the other hand $f_1 = f_2$ on $\Omega := \bigcup_{j=1}^{q}\big\{z : v^{k)}_{(f_1,H_j)}(z) > 0\big\}$. Hence, $(f_1, H_{j_0}) = (f_2, H_{j_0}) = 0$ on $\Omega \setminus f_1^{-1}(H_{i_0})$. So we have

$$\sum_{j=1, j\ne i_0}^{q} \overline{N}^{k)}_{f_1}(r, H_j) \le N\Big(r, v_{\frac{(f_1,H_{j_0})}{(f_1,H_{i_0})}}\Big) + \Big(\overline{N}\big(r, v_{(f_1,H_{i_0})}\big) - \overline{N}^{k)}\big(r, v_{(f_1,H_{i_0})}\big)\Big).$$

Thus, by the First and the Second Main Theorem, we have

$$\begin{aligned}(q-n-2)T_{f_1}(r) &\le \sum_{j=1, j\ne i_0}^{q} N_{n,f_1}(r, H_j) + o(T_{f_1}(r))\\ &\le n\sum_{j=1, j\ne i_0}^{q} N_{1,f_1}(r, H_j) + o(T_{f_1}(r))\end{aligned}$$

$$\leq \frac{nk}{k+1}\sum_{j=1,j\neq i_0}^{q}\overline{N}^{k)}_{f_1}(r,H_j)+\frac{n}{k+1}\sum_{j=1,j\neq i_0}^{q}N_{f_1}(r,H_j)+o(T_{f_1}(r))$$

$$\leq \frac{nk}{k+1}N\Big(r,v_{\frac{(f_1,H_{j_0})}{(f_1,H_{i_0})}}\Big)+\frac{nk}{k+1}\Big(\overline{N}\big(r,v_{(f_1,H_{i_0})}\big)-\overline{N}^{k)}\big(r,v_{(f_1,H_{i_0})}\big)\Big)$$
$$+\frac{(q-1)n}{k+1}T_{f_1}(r)+o(T_{f_1}(r))$$

$$\leq \frac{nk}{k+1}T_{\frac{(f_1,H_{j_0})}{(f_1,H_{i_0})}}(r)+\frac{nk}{(k+1)^2}N_{f_1}(r,H_{i_0})+\frac{(q-1)n}{k+1}T_{f_1}(r)+o\big(T_{f_1}(r)\big)$$
$$\leq \Big(\frac{nk}{k+1}+\frac{nk}{(k+1)^2}+\frac{(q-1)n}{k+1}\Big)T_{f_1}(r)+o\big(T_{f_1}(r)\big)$$

Thus, we get $(q-n-2)\leq \frac{nk}{k+1}+\frac{nk}{(k+1)^2}+\frac{(q-1)n}{k+1}\leq n+\frac{qn}{k}$. This contradicts any of the following cases:
i) $2\leq n\leq 3$, $q=3n+1$ and $k\geq 23n$,
ii) $4\leq n\leq 6$, $q=3n$ and $k\geq \frac{(6n-1)n}{n-3}$,
iii) $n\geq 7$, $q=3n-1$ and $k\geq \frac{(6n-4)n}{n-6}$.
Thus $\alpha_{ij}=1$ for all $1\leq i\neq j\leq 3n-1$.

By (19), for $i=3n-1, j\in\{1,\dots,3n-2\}$, without loss of generality, we may asssume that

$$\frac{(f_1,H_j)}{\big(f_1,H_{3n-1}\big)}=\frac{(f_2,H_j)}{\big(f_2,H_{3n-1}\big)},\ j=1,\dots,n: \tag{20}$$

For $1\leq s<v\leq 3$, denote by L_{sv} the set of all $j\in\{1,...,3n-2\}$ such that $\frac{(f_s,H_j)}{(f_s,H_{3n-1})}=\frac{(f_v,H_j)}{(f_v,H_{3n-1})}$. By (19) we have $L_{12}\cup L_{23}\cup L_{13}=\{1,...,3n-2\}$. So by Dirichlet we have that one of the three sets contains at least n different indices, which are, without loss of generality, $j=1,...,n$, which proves (20).

We choose homogeneous coordinates $(\omega_0:\cdots:\omega_n)$ on $\mathbb{C}P^n$ with $H_j=\{\omega_j=0\}$ $(1\leq j\leq n)$, $H_{3n-1}=\{\omega_0=0\}$ and take reduced representations: $f_1=(f_{1_0}:\cdots:f_{1_n})$, $f_2=(f_{2_0}:\cdots:f_{2_n})$. Then by (20) we have

$$\begin{cases}\dfrac{f_{1_j}}{f_{1_0}}=\dfrac{f_{2_j}}{f_{2_0}}\\ (j=1,\dots,n)\end{cases}\quad\Rightarrow\quad \frac{f_{1_0}}{f_{2_0}}=\cdots=\frac{f_{1_n}}{f_{2_n}}\ \Rightarrow f_1\equiv f_2.$$

This is a contradiction. Thus, for any case we have that f_1, f_2, f_3 can not be distinct. Hence, the Proof of Theorem 1 is complete. $\square$

Proof of Theorem 2. Assume that $\#F_k(\{H_j\}_{j=1}^q, f, 1) \geq 3$. Take arbitrarily three distinct mappings $f_1, f_2, f_3 \in F_k(\{H_j\}_{j=1}^q, f, 1)$. We have to prove that $f_s \times f_v : \mathbb{C}^m \longrightarrow \mathbb{C}P^n \times \mathbb{C}P^n$ is linearly degenerate for all $1 \leq s < v \leq 3$.

Denote by Q the set which contains all indices $j \in \{1, ..., q\}$ satisfing $\Phi^l\big(F_{1c}^j, F_{2c}^j, F_{3c}^j\big) \not\equiv 0$ for some $c \in \mathcal{C}$. We distinguish between the two cases n odd and n even:

Case 1. If n is odd, then $q = \frac{5(n+1)}{2}$.
We now pove that: $Q = \emptyset$. (21)

Indeed, otherwise there exist $j_0 \in Q$. Then by Lemma 2 (with $A = \emptyset, p = 1$) we have

$$2 \sum_{j=1, j\neq j_0}^{q} \overline{N}_{f_i}^{k)}(r, H_j) \leq \frac{k+2}{k+1} T(r) + o(T(r)),\ i = 1, 2, 3\ .$$

(note that $N_{0,f_i}^{k)}(r, H_{j_0}) = 0$). On the other hand, by Lemma 3 we have

$$2 \sum_{j=1, j\neq j_0}^{q} \overline{N}_{f_i}^{k)}(r, H_j) + o(T_{f_i}(r)) \geq \frac{2[(q-n-2)(k+1)-(q-1)n]}{nk} T_{f_i}(r)\ ,\ i = 1, 2, 3\ .$$

Hence, we have

$$((2q-2n-4)(k+1) - 2(q-1)n)\, T_{f_i}(r) \leq \frac{(k+2)nk}{k+1} T(r) + o(T(r))\ , i = 1, 2, 3,$$

which implies

$$\begin{aligned}\Big((2q-2n-4)(k+1) - 2(q-1)n\Big) T(r) &\leq \frac{3(k+2)nk}{k+1} T(r) + o(T(r)) \\ &\leq 3n(k+1)T(r) + o(T(r)).\end{aligned}$$

Hence, we obtain

$$(2q-2n-4)(k+1) - 2(q-1)n \leq 3n(k+1)$$

implying

$$k+1 \leq (5n+3)n.$$

This is a contradiction. Thus, we get (21).

Case 2. If n is even, then $q = \frac{5n+4}{2}$.

We now prove that $\#Q \leq 1$. (22)

Indeed, suppose that this assertion does not hold, then there exist two distinct indices $j_0, j_1 \in Q$. By Lemma 2 (with $A = \emptyset, p = 1$) we have

$$2 \sum_{j=1, j\neq j_0}^{q} \overline{N}_{f_i}^{k)}(r, H_j) \leq \frac{k+2}{k+1} T(r) + o(T(r)) \qquad , \ i = 1, 2, 3,$$

which implies that, for i=1,2,3

$$\begin{aligned} 2 \sum_{j=1, j\neq j_0}^{q} \left(\overline{N}_{f_i}^{k)}(r, H_j) - \frac{1}{n} N_{n, f_i}^{k)}(r, H_j) \right) &\leq \frac{k+2}{k+1} T(r) + o(T(r)) \\ -\frac{2}{n} \sum_{j=1, j\neq i_0}^{q} N_{n, f_i}^{k)}(r, H_j) \ , \ i &= 1, 2, 3. \end{aligned}$$

Hence, we get

$$\begin{aligned} 2 \sum_{j=1, j\neq j_0}^{q} \sum_{i=1}^{3} \left(\overline{N}_{f_i}^{k)}(r, H_j) - \frac{1}{n} N_{n, f_i}^{k)}(r, H_j) \right) &\leq \frac{3(k+2)}{k+1} T(r) + o(T(r)) \\ &-\frac{2}{n} \sum_{j=1, j\neq j_0}^{q} \sum_{i=1}^{3} N_{n, f_i}^{k)}(r, H_j), (23) \end{aligned}$$

By Lemma 3 (with $q = \frac{5n+4}{2}$), we have

$$2 \sum_{j=1, j\neq j_0}^{q} N_{n, f_i}^{k)}(r, H_j) + o(T_{f_i}(r)) \geq \frac{3n(k+1) - (5n+2)n}{k} T_{f_i}(r) \ , \ i = 1, 2, 3.$$

Hence, we have

$$\frac{2}{n} \sum_{j=1, j\neq j_0}^{q} \sum_{i=1}^{3} N_{n, f_i}^{k)}(r, H_j) + o(T(r)) \geq \frac{3n(k+1) - (5n+2)n}{nk} T(r) \qquad (24)$$

By (23) and (24) we have

$$\begin{aligned} 2 \sum_{j=1, j\neq j_0}^{q} \sum_{i=1}^{3} \left(\overline{N}_{f_i}^{k)}(r, H_j) - \frac{1}{n} N_{n, f_i}^{k)}(r, H_j) \right) &\leq \frac{(5n+2)n(k+1) - 3n}{nk(k+1)} T(r) + o(T(r)) \\ &\leq \frac{5n+2}{k} T(r) + o(T(r)). \end{aligned}$$

On the other hand, we obtain

$$\overline{N}^{k)}_{f_i}(r,H_j)-\ \frac{1}{n}N^{k)}_{n,f_i}(r,H_j)\geq 0 \text{ for all } i\in\{1,2,3\}, j\in\{1,...,q\}.$$

Hence, we get

$$\sum_{i=1}^{3}\left(\overline{N}^{k)}_{f_i}(r,H_j)-\ \frac{1}{n}N^{k)}_{n,f_i}(r,H_j)\right)\ \leq\frac{5n+2}{k}T(r)+o(T(r))\ ,\ j\in\{1,...,q\}\backslash\{j_0\}.$$

In particular, we get

$$\sum_{i=1}^{3}\left(\ \overline{N}^{k)}_{f_i}(r,H_{j_1})-\ \frac{1}{n}N^{k)}_{n,f_i}(r,H_{j_1})\right)\ \leq\frac{5n+2}{k}T(r)+o(T(r)) \tag{25}$$

Set $A_i:=\{z\in\mathbb{C}^m : v_{(f_i,H_{j_1})}(z)=1\}$ for $i=1,2,3$. For each $i\in\{1,2,3\}$, we have $\overline{A}_i\setminus A_i\subseteq \operatorname{sing} f_i^{-1}(H_{j_1})$. Indeed, otherwise there existed $a\in\left(\overline{A}_i\setminus A_i\right)\cap \operatorname{reg} f_i^{-1}(H_{j_1})$. Then $p_0:=v_{(f_i,H_{j_1})}(a)\geq 2$. Since a is a regular point of $f_i^{-1}(H_{j_1})$ we can choose nonzero holomorphic functions h and u on a neighborhood U of a such that $\mathrm{d}h$ and u have no zeroes and $(f_i,H_{j_1})\equiv h^{p_0}u$ on U. Since $a\in\overline{A}_i$ there exists $b\in A_i\cap U$. Then, we get $1=v_{(f_i,H_{j_1})}(b)=v_{h^{p_0}u}(b)=p_0\geq 2$. This is a contradiction. Thus, we get that $\overline{A}_i\setminus A_i\subseteq \operatorname{sing} f_i^{-1}(H_{j_1})$.

Set $B:=A_1\cup A_2\cup A_3$. Then $\overline{B}\setminus B\subseteq\bigcup_{i=1}^{3}\operatorname{sing} f_i^{-1}(H_{j_1})$. This means that $\overline{B}\setminus B$ is included in an analytic set of codimension ≥ 2. So we have

$$(n-1)\bar{\bar{N}}(r,\overline{B})\leq\sum_{i=1}^{3}\left(n\ \overline{N}^{k)}_{f_i}(r,H_{j_1})-\ N^{k)}_{n,f_i}(r,H_{j_1})\right).$$

By (25) we have

$$\bar{\bar{N}}(r,\overline{B})\leq\frac{(5n+2)n}{(n-1)k}T(r)+o(T(r)),$$

where we note that $n\geq 2$, since n is even. It is clear that
$min\{v^{k)}_{(f_1,H_{j_1})},2\}=min\{v^{k)}_{(f_2,H_{j_1})},2\}=min\{v^{k)}_{(f_3,H_{j_1})},2\}$ on $\mathbb{C}^m\backslash\overline{B}(\subseteq\mathbb{C}^m\diagdown B)$.

By Lemma 2 (with $A = \overline{B}$, $p = 2$) we have

$$2\sum_{j=1,j\neq j_1}^{q} \overline{N}^{k)}_{f_i}(r,H_j) + \overline{N}^{k)}_{f_i}(r,H_{j_1}) \le \frac{k+2}{k+1}T(r) + 4\overline{N}(r,\overline{B}) + o(T(r))$$
$$\le \left(\frac{k+2}{k+1} + \frac{4(5n+2)n}{(n-1)k}\right)T(r) + o(T(r)) \ , \quad (26)$$

(note that $j_1 \in Q$). By Lemma 3 we have

$$\sum_{j=1,j\neq j_1}^{q} \overline{N}^{k)}_{f_i}(r,H_j) + o(T_{f_i}(r)) \ge \frac{(q-n-2)(k+1)-(q-1)n}{nk}T_{f_i}(r) \text{ , and}$$
$$\sum_{j=1}^{q} \overline{N}^{k)}_{f_i}(r,H_j) + o(T_{f_i}(r)) \ge \frac{(q-n-1)(k+1)-qn}{nk}T_{f_i}(r) \ .$$

Consequently, we obtain

$$2\sum_{j=1,j\neq j_1}^{q} \overline{N}^{k)}_{f_i}(r,H_j) + \overline{N}^{k)}_{f_i}(r,H_{j_1}) + o(T_{f_i}(r)) \ge \frac{(2q-2n-3)(k+1)-(2q-1)n}{nk}T_{f_i}(r) \quad (27)$$

By (26) and (27) we have

$$\frac{(2q-2n-3)(k+1)-(2q-1)n}{nk}T_{f_i}(r) \le \left(\frac{k+2}{k+1} + \frac{4(5n+2)n}{(n-1)k}\right)T(r) + o(T(r)),$$

which implies

$$((3n+1)(k+1)-(5n+3)n)\,T(r) \le \left(\frac{3nk(k+2)}{k+1} + \frac{12(5n+2)n^2}{(n-1)}\right)T(r) + o(T(r))$$
$$\le (3n(k+1) + \frac{12(5n+2)n^2}{(n-1)})T(r) + o(T(r)),$$

and, hence,

$$k+1 \le (5n+3)n + \frac{12(5n+2)n^2}{(n-1)}.$$

This contradicts $k \ge (65n+171)n$, $n \ge 2$. Hence, we have $\#Q \le 1$. So we get (22).

By (21) and (22) we have $\#(\{1,...,q\} \setminus Q) \geq q-1$. Without loss of generality we may assume that $1,...,q-1 \notin Q$. For any $j \in \{1,\ldots,q-1\}$ we have

$\Phi^l\big(F^j_{1c}, F^j_{2c}, F^j_{3c}\big) \equiv 0$ for all $c \in \mathcal{C}$, $l \in \{1,...,m\}$.

On the other hand, $\mathcal{C}$ is dense in $\mathbb{C}^{n+1}$. Hence, we get that $\Phi^l\big(F^j_{1c}, F^j_{2c}, F^j_{3c}\big) \equiv 0$ for all $c \in \mathbb{C}^{n+1} \setminus \{0\}$, $l \in \{1,...,m\}$, $j \in \{1,\ldots,q-1\}$. In particular (for $H_c = H_i$), we get

$$\Phi^l\left(\frac{(f_1,H_j)}{(f_1,H_i)},\ \frac{(f_2,H_j)}{(f_2,H_i)},\ \frac{(f_3,H_j)}{(f_3,H_i)}\right) \equiv 0$$

for all $1 \leq i \neq j \leq q-1$, $\quad l \in \{1,\ldots,m\}$.

For each $1 \leq i \neq j \leq q-1$, by Theorem 2.2, there exists a constant α_{ij} such that

$$\frac{(f_2,H_j)}{(f_2,H_i)} = \alpha_{ij}\frac{(f_1,H_j)}{(f_1,H_i)} \text{ or } \frac{(f_3,H_j)}{(f_3,H_i)} = \alpha_{ij}\frac{(f_1,H_j)}{(f_1,H_i)} \text{ or } \frac{(f_3,H_j)}{(f_3,H_i)} = \alpha_{ij}\frac{(f_2,H_j)}{(f_2,H_i)} .$$

We now prove that

$\alpha_{ij} = 1$ for all $1 \leq i \neq j \leq q-1$. (28)

Indeed, if there exists $\alpha_{i_0j_0} \neq 1$, without loss of generality, we may assume that $\frac{(f_2,H_{j_0})}{(f_2,H_{i_0})} = \alpha_{i_0j_0}\frac{(f_1,H_{j_0})}{(f_1,H_{i_0})}$. On the other hand, we have $f_1 = f_2$ on $D := \bigcup_{j=1}^{q}\big\{z : v^{k)}_{(f_1,H_j)} > 0\big\}$. Hence, we get $(f_1,H_{j_0}) = (f_2,H_{j_0}) = 0$ on $D \setminus f_1^{-1}(H_{i_0})$. So we have

$$\sum_{j=1,j\neq i_0}^{q} \overline{N}^{k)}_{f_1}(r,H_j) \leq N\Big(r, v_{\frac{(f_1,H_{j_0})}{(f_1,H_{i_0})}}\Big) + \Big(\overline{N}\big(r, v_{(f_1,H_{i_0})}\big) - \overline{N}^{k)}\big(r, v_{(f_1,H_{i_0})}\big)\Big).$$

Thus, by the First and the Second Main Theorem, we have

$$\begin{aligned}(q-n-2)T_{f_1}(r) &\leq \sum_{j=1,j\neq i_0}^{q} N_{n,f_1}(r,H_j) + o(T_{f_1}(r))\\ &\leq n\sum_{j=1,j\neq i_0}^{q} N_{1,f_1}(r,H_j) + o(T_{f_1}(r))\end{aligned}$$

$$\leq \frac{nk}{k+1}\sum_{j=1,j\neq i_0}^{q}\overline{N}^{k)}_{f_1}(r,H_j)+\frac{n}{k+1}\sum_{j=1,j\neq i_0}^{q}N_{f_1}(r,H_j)+o(T_{f_1}(r))$$

$$\leq \frac{nk}{k+1}N\Big(r,v_{\frac{(f_1,H_{j_0})}{(f_1,H_{i_0})}}\Big)+\frac{nk}{k+1}\Big(\overline{N}\big(r,v_{(f_1,H_{i_0})}\big)-\overline{N}^{k)}\big(r,v_{(f_1,H_{i_0})}\big)\Big)$$
$$+\frac{(q-1)n}{k+1}T_{f_1}(r)+o(T_{f_1}(r))$$

$$\leq \frac{nk}{k+1}T_{\frac{(f_1,H_{j_0})}{(f_1,H_{i_0})}}(r)+\frac{nk}{(k+1)^2}N_{f_1}(r,H_{i_0})+\frac{(q-1)n}{k+1}T_{f_1}(r)+o\big(T_{f_1}(r)\big)$$
$$\leq \Big(\frac{nk}{k+1}+\frac{nk}{(k+1)^2}+\frac{(q-1)n}{k+1}\Big)T_{f_1}(r)+o\big(T_{f_1}(r)\big)$$

Thus, we have $(q-n-2)\leq \frac{nk}{k+1}+\frac{nk}{(k+1)^2}+\frac{(q-1)n}{k+1}\leq n+\frac{nq}{k}$.
This contradicts $q=\left[\frac{5(n+1)}{2}\right], k\geq (65n+171)n$. Thus, we get that $\alpha_{ij}=1$ for all $1\leq i\neq j\leq q-1$.

For $1\leq s<v\leq 3$, denote by L_{sv} the set of all $j\in\{1,...,q-2\}$ such that $\frac{(f_s,H_j)}{(f_s,H_{q-1})}=\frac{(f_v,H_j)}{(f_v,H_{q-1})}$. By (28) , we have that $L_{12}\cup L_{23}\cup L_{13}=\{1,...,q-2\}$.

If there exists some $L_{sv}=\emptyset$, without loss of generality, we may assume that $L_{13}=\emptyset$. Then $L_{12}\cup L_{23}=\{1,...,q-2\}$. Since $q=\left[\frac{5(n+1)}{2}\right]$ we have that $\#L_{12}\geq n$ or $\#L_{23}\geq n$. We may assume that $\#L_{12}\geq n$, and furthermore $1,...,n\in L_{12}$. Then $\frac{(f_1,H_j)}{(f_1,H_{q-1})}=\frac{(f_2,H_j)}{(f_2,H_{q-1})}$ for all $j\in\{1,...,n\}$, so $f_1\equiv f_2$ (as in the proof of Theorem 1). This is a contradiction.

Thus, we have $L_{sv}\neq\emptyset$ for all $1\leq s<v\leq 3$. Then for any $1\leq s<v\leq 3$, there exists $j\in\{1,...,q-2\}$ such that $\frac{(f_s,H_j)}{(f_s,H_{q-1})}=\frac{(f_v,H_j)}{(f_v,H_{q-1})}$. Hence, we finally get that $f_s\times f_v:\mathbb{C}^m\longrightarrow \mathbb{C}P^n\times\mathbb{C}P^n$ is linearly degenerate. We thus have completed the proof of Theorem 2. □

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