A SECOND MAIN THEOREM FOR MOVING HYPERSURFACE TARGETS

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Abstract

In this paper, we prove a Second Main Theorem for algebraically non-degenerate meromorphic maps of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and slowly moving hypersurfaces targets $Q_j \subset \mathbb{C}P^n$, $j = 1, \ldots, q$ ($q \geq n+2$) in (weakly) general position. This generalizes the Second Main Theorem for fixed hypersurface targets in general position, obtained by M. Ru in [20]. We also introduce a truncation, with an explicit estimate of the truncation level, into this Second Main Theorem with moving targets, thus generalizing the main result of An-Phuong [1].

1 Introduction

For $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$, we set $\|z\| = \Big(\sum_{j=1}^{m} |z_j|^2\Big)^{1/2}$ and define

$$B(r) = \{z \in \mathbb{C}^m : \|z\| < r\}, \quad S(r) = \{z \in \mathbb{C}^m : \|z\| = r\},$$

$$d^c = \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial), \quad \mathcal{V} = \big(dd^c\|z\|^2\big)^{m-1}, \ \sigma = d^c\log\|z\|^2 \wedge \big(dd^c\log\|z\|\big)^{m-1}.$$

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Let L be a positive integer or $+\infty$ and ν be a divisor on $\mathbb{C}^m$. Set $|\nu| = \overline{\{z : \nu(z) \neq 0\}}$. We define the counting function of ν by

$$N_\nu^{(L)}(r) := \int_1^r \frac{n^{(L)}(t)}{t^{2m-1}} dt \quad (1 < r < +\infty),$$

where

$$n^{(L)}(t) = \int_{|\nu| \cap B(t)} \min\{\nu, L\} \cdot \mathcal{V} \quad \text{for} \quad m \geq 2 \text{ and}$$

$$n^{(L)}(t) = \sum_{|z| \leqslant t} \min\{\nu(z), L\} \qquad \text{for} \quad m = 1.$$

Let F be a nonzero holomorphic function on $\mathbb{C}^m$. For a set $\alpha = (\alpha_1, \ldots, \alpha_m)$ of nonnegative integers, we set $|\alpha| := \alpha_1 + \cdots + \alpha_m$ and $D^\alpha F := \dfrac{\partial^{|\alpha|}}{\partial^{\alpha_1} z_1 \cdots \partial^{\alpha_m} z_m} \cdot$ We define the zero divisor ν_F of F by

$$\nu_F(z) = \max\big\{p : D^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < p\big\}.$$

Let φ be a nonzero meromorphic function on $\mathbb{C}^m$. The zero divisor ν_φ of φ is defined as follows: For each $a \in \mathbb{C}^m$, we choose nonzero holomorphic functions F and G on a neighborhood U of a such that $\varphi = \dfrac{F}{G}$ on U and $\dim\big(F^{-1}(0) \cap G^{-1}(0)\big) \leqslant m - 2$, then we put $\nu_\varphi(a) := \nu_F(a)$.

Set $N_\varphi^{(L)}(r) := N_{\nu_\varphi}^{(L)}(r)$. For brevity we will omit the character $^{(L)}$ in the counting function if $L = +\infty$.

Let f be a meromorphic map of $\mathbb{C}^m$ into $\mathbb{C}P^n$. For arbitrary fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ of $\mathbb{C}P^n$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, which means that each f_i is a holomorphic function on $\mathbb{C}^m$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ outside the analytic set $\{z : f_0(z) = \cdots = f_n(z) = 0\}$ of codimension ≥ 2. Set $\|f\| = \max\{|f_0|, \ldots, |f_n|\}$.

The characteristic function of f is defined by

$$T_f(r) := \int_{S(r)} \log\|f\|\sigma - \int_{S(1)} \log\|f\|\sigma, \quad 1 < r < +\infty.$$

For a meromorphic function φ on $\mathbb{C}^m$, the characteristic function $T_\varphi(r)$ of φ is defined by considering φ as a meromorphic map of $\mathbb{C}^m$ into $\mathbb{C}P^1$. We have the following Jensen's formula :

$$N_\varphi(r) - N_{\frac{1}{\varphi}}(r) = \int\limits_{S(r)} \log|\varphi|\sigma - \int\limits_{S(1)} \log|\varphi|\sigma.$$

Let f be a nonconstant meromorphic map of $\mathbb{C}^m$ into $\mathbb{C}P^n$. We say that a meromorphic function φ on $\mathbb{C}^m$ is "small" with respect to f if $T_\varphi(r) = o(T_f(r))$ as $r \to \infty$ (outside a set of finite Lebesgue measure).

Denote by $\mathcal{K}_f$ the set of all "small" (with respect to f) meromorphic functions on $\mathbb{C}^m$. By Theorem 5.2.29 of [16] or by Corollary 5.7 in [11] we easily get that any rational expression of functions in $\mathcal{K}_f$ is still "small" (with respect to f), in particular $\mathcal{K}_f$ is a field.

For a homogeneous polynomial $Q \in \mathcal{K}_f[x_0, \ldots, x_n]$ of degree $d \geq 1$ with $Q(f_0, \ldots, f_n) \not\equiv 0$, we define

$$N_f^{(L)}(r, Q) := N^{(L)}_{Q(f_0,\ldots,f_n)}(r) \text{ and } \delta_f(Q) = \liminf_{r\to\infty} \Big(1 - \frac{N_f(r,Q)}{d \cdot T_f(r)}\Big).$$

Denote by $Q(z)$ the homogeneous polynomial over $\mathbb{C}$ obtained by evaluating the coefficients of Q at a specific point $z \in \mathbb{C}^m$ in which all coefficient functions of Q are holomorphic.

For a positive integer d, we set

$$\mathcal{T}_d := \big\{(i_0, \ldots, i_n) \in \mathbb{N}_0^{n+1} : i_0 + \cdots + i_n = d\big\}.$$

Let

$$Q_j = \sum_{I \in \mathcal{T}_{d_j}} a_{jI} x^I \quad (j = 1, \ldots, q)$$

be homogeneous polynomials in $\mathcal{K}_f[x_0, \ldots, x_n]$ with $\deg Q_j = d_j \geq 1$, where $x^I = x_0^{i_0} \cdots x_n^{i_n}$ for $x = (x_0, \ldots, x_n)$ and $I = (i_0, \ldots, i_n)$. Denote by $\mathcal{K}_{\{Q_j\}_{j=1}^q}$ the field over $\mathbb{C}$ of all meromorphic functions on $\mathbb{C}^m$ generated by $\big\{a_{jI} : I \in \mathcal{T}_{d_j}, j \in \{1, \ldots, q\}\big\}$. It is clearly a subfield of $\mathcal{K}_f$. Denote by $\tilde{\mathcal{K}}_{\{Q_j\}_{j=1}^q} \subset \mathcal{K}_{\{Q_j\}_{j=1}^q}$ the subfield generated by all quotients $\big\{\frac{a_{jI_1}}{a_{jI_2}} : a_{jI_2} \neq 0, I_1, I_2 \in \mathcal{T}_{d_j}; j \in \{1, \ldots, q\}\big\}$. We say that f is algebraically nondegenerate over

$\mathcal{K}_{\{Q_j\}_{j=1}^q}$ (respectively over $\tilde{\mathcal{K}}_{\{Q_j\}_{j=1}^q}$) if there is no nonzero homogeneous polynomial $Q \in \mathcal{K}_{\{Q_j\}_{j=1}^q}[x_0, \ldots, x_n]$ (respectively $Q \in \tilde{\mathcal{K}}_{\{Q_j\}_{j=1}^q}[x_0, \ldots, x_n]$) such that $Q(f_0, \ldots, f_n) \equiv 0$.

We say that a set $\{Q_j\}_{j=1}^q$ $(q \geq n+1)$ of homogeneous polynomials in $\mathcal{K}_f[x_0, \ldots, x_n]$ is admissible (or in (weakly) general position) if there exists $z \in \mathbb{C}^m$ in which all coefficient functions of all Q_j, $j = 1, ..., q$ are holomorphic and such that for any $1 \leqslant j_0 < \cdots < j_n \leqslant q$ the system of equations

$$\begin{cases} Q_{j_i}(z)(x_0, \ldots, x_n) = 0 \\ \qquad 0 \leqslant i \leqslant n \end{cases} \tag{1.1}$$

has only the trivial solution $(x_0, \ldots, x_n) = (0, \ldots, 0)$ in $\mathbb{C}^{n+1}$. We remark that in this case this is true for the generic $z \in \mathbb{C}^m$.

As usual, by the notation "$\| P$" we mean the assertion P holds for all $r \in [1, +\infty)$ excluding a Borel subset E of $(1, +\infty)$ with $\int_E dr < +\infty$.

Main Theorem. *Let f be a nonconstant meromorphic map of $\mathbb{C}^m$ into $\mathbb{C}P^n$. Let $\left\{Q_j\right\}_{j=1}^q$ be an admissible set of homogeneous polynomials in $\mathcal{K}_f[x_0, \ldots, x_n]$ with $\deg Q_j = d_j \geq 1$. Assume that f is algebraically nondegenerate over $\tilde{\mathcal{K}}_{\{Q_j\}_{j=1}^q}$. Then for any $\varepsilon > 0$, there exist positive integers L_j $(j = 1, \ldots, q)$, depending only on n, ε and d_j $(j = 1, \ldots, q)$ in an explicit way such that*

$$\|(q - n - 1 - \varepsilon) T_f(r) \leqslant \sum_{j=1}^{q} \frac{1}{d_j} N_f^{(L_j)}(r, Q_j).$$

We note that, for fixed hypersurface targets, in 1979, Shiffman [22] conjectured that if f is an algebraically nondegenerate holomorphic map of $\mathbb{C}$ into $\mathbb{C}P^n$ and $D_1, \cdots, D_q$ are hypersurfaces in $\mathbb{C}P^n$ in general position, then $\sum_{j=1}^q \delta_f(D_j) \leqslant n+1$. This conjecture was proved by Ru [20] in 2004, and recently even generalized by Ru [21] to fixed hypersurface sections of projective varieties in general position. As a corollary of the Main Theorem we get the generalization of his result in [20] for moving targets.

Corollary 1.1 (Shiffman conjecture for moving hypersurfaces)**.** *Under the same assumption as in the Main theorem, we have*

$$\sum_{j=1}^{q} \delta_f(Q_j) \leqslant n+1.$$

We also note that for the case of moving hyperplanes ($d_1 = \cdots = d_q = 1$), and multiplicities which are not truncated, the above theorem was first proved by Ru and Stoll in 1991 [17]. In 2002, Tu [25] introduced a truncation into the Second Main Theorem of Ru-Stoll, but the truncation level is is not estimated. Furthermore, after the first version [7] of this paper was published, in which the truncation level was not estimated, neither, An-Phuong [1] gave a truncation with an explicit estimate for the Second Main Theorem for fixed hypersurfaces. So our Main Theorem, now also with an explicit estimate of the truncation level, is also a generalization of their result to moving hypersurfaces. In the special case of fixed hypersurfaces our estimate for the truncation is still slightly better, but, at least in the case when all hypersurfaces are of the same degree, still of the same order than theirs.

Proposition 1.2. *With the notation of our Main Theorem, we have*

$$L_j \leqslant \frac{d_j \cdot \binom{n+N}{n} t_{p_0+1} - d_j}{d} + 1\,,$$

where d is the least common multiple of the d_j's and

$$N = d \cdot [2(n+1)(2^n - 1)(nd+1)\epsilon^{-1} + n + 1]\,,$$

$$p_0 = \Big[\frac{\Big(\binom{n+N}{n}^2 . \binom{q}{n} - 1\Big). \log\Big(\binom{n+N}{n}^2 . \binom{q}{n}\Big)}{\log\big(1 + \frac{\epsilon}{2\binom{n+N}{n}N}\big)} + 1\Big]^2,$$

$$\textit{and} \quad t_{p_0+1} < \left(\binom{n+N}{n}^2 . \binom{q}{n} + p_0\right)^{\binom{n+N}{n}^2 . \binom{q}{n} - 1},$$

where we denote $[x] := \max\{k \in \mathbb{Z} : k \leqslant x\}$ for a real number x. Furthermore, in the case of fixed hypersurfaces ($Q_j \in \mathbb{C}[x_0, \ldots, x_n], j = 1, \ldots, q$), we have $t_p = 1$ for all positive integers p, so we get a better estimate:

$$L_j \leqslant \frac{d_j \cdot \binom{n+N}{n} - d_j}{d} + 1.$$

Remark 1.3. *The Main Theorem holds, more generally, for an admissible set of polynomials $\{Q_j\}_{j=1}^q$ such that only the quotients $\{\frac{a_{jI_1}}{a_{jI_2}} : a_{jI_2} \neq 0, I_1, I_2 \in \mathcal{T}_{d_j}; j \in \{1, \ldots, q\}\}$ lie in $\mathcal{K}_f$, under the condition that one replaces the $N_f^{(L)}(r, Q_j)$ by $N_f^{(L)}(r, \frac{1}{a_{jI_2}} Q_j)$, where $a_{jI_2} \neq 0$ can be any nonzero* *coefficient of Q_j, $j = 1, ..., q$. This follows immediately from the Main Theorem, applied to the set of polynomials $\left\{\tilde{Q}_j\right\}_{j=1}^{q}$, where $\tilde{Q}_j := \frac{1}{a_{jI_2}} Q_j$. For more details, see the beginning of section 4.*

The proof of our Main Theorem (including the one of Proposition 1.2) consists of three main parts, in which the second and the third one are considerably more complicated than this can be done for fixed hypersurfaces with their notion of general position:

In the first part (chapter 4 until equation (4.14)) we use the idea of Corvaja-Zannier [6] and Ru [20] to estimate $log \prod_{j=1}^{q} |Q_j(f)|$. However, we have to pass many difficulties which come both from the facts that the concept "in general position" in our paper is more general than in Corvaja-Zannier's and Ru's paper and that the field $\mathcal{K}_f$ is not algebraically closed in general, so we cannot use any more Hilbert's Nullstellensatz. Instead we have to use explicit results on resultants respectively discriminant varieties for universal families of configurations of q hypersurfaces in $\mathbb{C}P^n$, generalizing, among others, considerably Hilbert's Nullstellensatz (see [14], chapter IX). This allows us to deal with such hypersurfaces with "variable" coefficients, namely in $\mathcal{K}_f$, but by specialization to the fibers to have nevertheless complex solutions of these configurations of hypersurfaces. Another problem related to the fact that $\mathcal{K}_f$ is not algebraically closed in general is that the proof of the fact that admissible families of polynomials in $\mathcal{K}_f[x_0, ..., x_n]$ give regular families does not follow any more directly from Hilbert's Nullstellensatz, but needs another time resultants, as well as results on parameter systems in Cohen-Macauley rings.

In the second part (up to equation (4.22), we estimate the "error term" of equation (4.14), relating it moreover to a Wronskian, which will become crucial to give the truncation in the third part. It is in particular here where generalizing the coefficients from constants to meromorphic functions (although slowly growing ones) complicates substantially the analysis, especially with respect to the Wronskians and the Lemma of Logarithmic Derivative. Here we have to introduce technics known from Value Distribution Theory of moving hyperplanes (which we take from Shirosaki [23]), and to adopt them from the hyperplane to the hypersurface case. Another complication compared to the moving hyperplane case is that we cannot use once and for all reduced representations for the coefficient functions of the polynomials giving the moving hypersurfaces, which needs a special care while we take pointwise maxima or minima of their norms and while we estimate error terms. It is only at the end of the proof when we use a Lemma of Logarithmic Derivative for wronskians, where we pass to a reduced representation of a particular meromorphic map from $\mathbb{C}^m$ with monomial coefficients in the components of f and the coefficients of the $Q_j, j = 1, \dots q$. We finally remark that in this part, instead using the lemma of logarithmic derivative, we also could have used Theorem 2.3 of Ru [19].

In the third part, truncation is obtained. Here the concept "resultants of homogenous polynomials" and Wronskians are used again, now to estimate the corresponding divisors. The use of this tool, which is not necessary in the case of fixed hypersurfaces, is necessary in the case of moving hypersurfaces because of our very general notion of general position, in order to control what happens over the divisor where the resultant vanishes, this means where the hypersurfaces are not in general position.

We finally remark that we prefered to prove our result right away for meromorphic maps rather than only for the most important special case, namely entire holomorphic curves, since this proof is only around two pages longer than the one we could have given for entire curves.

2 Some lemmas

We first recall some classical results on resultants, see Lang [14], section IX.3, for the precise definition, the existence and for the principal properties of resultants, as well as Eremenko-Sodin [8], page 127: Let $\{Q_j\}_{j=0}^n$ be a set of homogeneous polynomials of common degree $d \geq 1$ in $\mathcal{K}_f[x_0, \dots, x_n]$

$$Q_j = \sum_{I \in \mathcal{T}_d} a_{jI} x^I, \quad a_{jI} \in \mathcal{K}_f \quad (j = 0, \dots, n).$$

Let $T = (\dots, t_{kI}, \dots)$ $(k \in \{0, \dots, n\}, I \in \mathcal{T}_d)$ be a family of variables. Set

$$\widetilde{Q}_j = \sum_{I \in \mathcal{T}_d} t_{jI} x^I \in \mathbb{Z}[T, x], \quad j = 0, \dots, n.$$

Let $\widetilde{R} \in \mathbb{Z}[T]$ be the resultant of $\widetilde{Q}_0, \dots, \widetilde{Q}_n$. This is a polynomial in the variables $T = (\dots, t_{kI}, \dots)$ $(k \in \{0, \dots, n\}, I \in \mathcal{T}_d)$ with integer coefficients, such that the condition $\widetilde{R}(T) = 0$ is necessary and sufficient for the existence of a nontrivial solution $(x_0, \dots, x_n) \neq (0, \dots, 0)$ in $\mathbb{C}^{n+1}$ of the system of equations

$$\begin{cases} \widetilde{Q}_j(T)(x_0, \ldots, x_n) = 0 \\ 0 \leqslant i \leqslant n \end{cases} . \tag{2.1}$$

From equations (2.1) and (1.1) is follows immediately that if

$$\{Q_j = \widetilde{Q}_j(a_{jI})(x_0, \ldots, x_n)\,,\ j = 0, \ldots, n\}$$

is an admissible set,

$$R := \widetilde{R}(\ldots, a_{kI}, \ldots) \not\equiv 0\,. \tag{2.2}$$

Furthermore, since $a_{kI} \in \mathcal{K}_f$, we have $R \in \mathcal{K}_f$. We finally will need the following result on resultants, which is contained in Theorem 3.4 in [14] (see also Eremenko-Sodin [8], page 127, for a similar result):

Proposition 2.1. *There exists a positive integer s and polynomials $\{\widetilde{b}_{ij}\}_{0\leqslant i,j\leqslant n}$ in $\mathbb{Z}[T,x]$, which are (without loss of generality) zero or homogenous in x of degree $s-d$, such that*

$$x_i^s \cdot \widetilde{R} = \sum_{j=0}^{n} \widetilde{b}_{ij}\widetilde{Q}_j \quad \textit{for all } i \in \{0, \ldots, n\}.$$

Let f be a nonconstant meromorphic map of $\mathbb{C}^m$ into $\mathbb{C}P^n$. Denote by $\mathcal{C}_f$ the set of all non-negative functions $h : \mathbb{C}^m \setminus A \longrightarrow [0, +\infty] \subset \overline{\mathbb{R}}$, which are of the form

$$\frac{|g_1| + \cdots + |g_k|}{|g_{k+1}| + \cdots + |g_l|}\,, \tag{2.3}$$

where $k, l \in \mathbb{N}$, $g_1, \ldots, g_l \in \mathcal{K}_f \setminus \{0\}$ and $A \subset \mathbb{C}^m$, which may depend on $g_1, \cdots, g_l$, is an analytic set of codimension at least two. By Jensen's formula and the First Main Theorem we have

$$\int_{S(r)} \log|\phi|\sigma = o(T_f(r)) \quad \text{as } r \to \infty$$

for $\phi \in \mathcal{K}_f \setminus \{0\}$. Hence, for any $h \in \mathcal{C}_f$, we have

$$\int_{S(r)} \log h\sigma = o(T_f(r)) \quad \text{as } r \to \infty.$$

It is easy to see that sums, products and quotients of functions in $\mathcal{C}_f$ are again in $\mathcal{C}_f$. We would like to point out that, in return, given any functions $g_1, \cdots, g_l \in \mathcal{K}_f \setminus \{0\}$, any expression of the form (2.3) is in fact a well defined function (with values in $[0, +\infty]$) outside an analytic subset A of codimension at least two, even though all the $g_1, \cdots, g_l$ can have common pole or zero divisors in codimension one.

Lemma 2.2. *Let $\{Q_j\}_{j=0}^n$ be a set of homogeneous polynomials of degree d in $\mathcal{K}_f[x_0, \ldots, x_n]$. Then there exists a function $h_1 \in \mathcal{C}_f$ such that, outside an analytic set of $\mathbb{C}^m$ of codimension at least two,*

$$\max_{j \in \{0,\ldots,n\}} |Q_j(f_0, \ldots, f_n)| \leqslant h_1 \cdot \|f\|^d.$$

If, moreover, this set of homogeneous polynomials is admissible, then there exists a nonzero function $h_2 \in \mathcal{C}_f$ such that, outside an analytic set of $\mathbb{C}^m$ of codimension at least two,

$$h_2 \cdot \|f\|^d \leqslant \max_{j \in \{0,\ldots,n\}} |Q_j(f_0, \ldots, f_n)|.$$

Proof. Assume that

$$Q_j = \sum_{I \in \mathcal{T}_d} a_{jI} x^I, \quad a_{jI} \in \mathcal{K}_f \quad (j = 0, \ldots, n).$$

We have, outside a proper analytic set of $\mathbb{C}^m$,

$$|Q_j(f_0, \ldots, f_n)| = \Big| \sum_{I \in \mathcal{T}_d} a_{jI} f^I \Big| \leqslant \sum_{I \in \mathcal{T}_d} |a_{jI}| \cdot \|f\|^d. \tag{2.4}$$

Set

$$h_1 := \sum_{j=0}^{n} \sum_{I \in \mathcal{T}_d} |a_{jI}|.$$

Then $h_1 \in \mathcal{C}_f$, since $a_{jI} \in \mathcal{K}_f$ $(j \in \{0, \ldots, n\}, I \in \mathcal{T}_d)$. By (2.4), we get

$$|Q_j(f_0, \ldots, f_n)| \leqslant |h_1| \cdot \|f\|^d \quad \text{for all } j \in \{0, \ldots, n\}.$$

So we have

$$\max_{j \in \{0,\ldots,n\}} |Q_j(f_0, \ldots, f_n)| \leqslant h_1 \cdot \|f\|^d. \tag{2.5}$$

All expressions in the last inequality are well defined and continuous (as functions with values in $[0,+\infty]$) outside analytic sets of codimension at least two. Since $\|f\|^d$ is a real-valued function which is zero only on an analytic subset of $\mathbb{C}^m$ of codimension at least two, this inequality still holds outside an analytic subset of $\mathbb{C}^m$ of codimension at least two.

In order to prove the second inequality, by Proposition 2.1 and its notations we have: There exists a positive integer s and polynomials $\{\widetilde{b}_{ij}\}_{0\leqslant i,j\leqslant n}$ in $\mathbb{Z}[T,x]$, zero or homogenous in x of degree $s-d$, such that

$$x_i^s \cdot \widetilde{R} = \sum_{j=0}^{n} \widetilde{b}_{ij}\widetilde{Q}_j \quad \text{for all } i \in \{0,\dots,n\}.$$

Moreover, $R = \widetilde{R}(\dots, a_{kI}, \dots) \not\equiv 0$. Set

$$b_{ij} = \widetilde{b}_{ij}\big((\dots, a_{kI}, \dots), (f_0, \dots, f_n)\big), \quad 0 \leqslant i, j \leqslant n.$$

Then, we get

$$f_i^s \cdot R = \sum_{j=0}^{n} b_{ij} \cdot Q_j(f_0, \dots, f_n) \quad \text{for all } i \in \{0,\dots,n\}.$$

So we have, outside a proper analytic set of $\mathbb{C}^m$:

$$\begin{aligned} |f_i^s \cdot R| &= \Big| \sum_{j=0}^{n} b_{ij} \cdot Q_j(f_0, \dots, f_n) \Big| \\ &\leqslant \sum_{j=0}^{n} |b_{ij}| \cdot \max_{k\in\{0,\dots,n\}} |Q_k(f_0, \dots, f_n)| \end{aligned} \tag{2.6}$$

for all $i \in \{0,\dots,n\}$. We write

$$b_{ij} = \sum_{I \in \mathcal{T}_{s-d}} \gamma_I^{ij} f^I, \quad \gamma_I^{ij} \in \mathcal{K}_f.$$

By (2.6), we get

$$|f_i^s \cdot R| \leqslant \sum_{\substack{0\leqslant j\leqslant n \\ I\in\mathcal{T}_{s-d}}} \big|\gamma_I^{ij}\big| \cdot \|f\|^{s-d} \cdot \max_{k\in\{0,\dots,n\}} |Q_k(f_0, \dots, f_n)|, \quad i \in \{0,\dots,n\}.$$

So

$$\frac{|f_i|^s}{\|f\|^{s-d}} \leqslant \sum_{\substack{0\leqslant j\leqslant n\\ I\in\mathcal{T}_{s-d}}} \left|\frac{\gamma_I^{ij}}{R}\right| \max_{k\in\{0,\dots,n\}} |Q_k(f_0,\dots,f_n)| \tag{2.7}$$

for all $i \in \{0, \dots, n\}$. Set

$$h_2 = \frac{1}{\sum_{i=0}^n \sum_{\substack{0\leqslant j\leqslant n\\ I\in\mathcal{T}_{s-d}}} \left|\frac{\gamma_I^{ij}}{R}\right|}.$$

Then $h_2 \in \mathcal{C}_f$, since $\gamma_I^{ij}, R \in \mathcal{K}_f$ and $R \not\equiv 0$. By (2.7) and since $\|f\|$ was the maximum norm, so $\|f\| = |f_i|$ for some $i = 0, \dots, n$ (which may depend on $z \in \mathbb{C}^m$), we have

$$h_2 \cdot \|f\|^d \leqslant \max_{j\in\{0,\dots,n\}} |Q_j(f_0,\dots,f_n)|\,. \tag{2.8}$$

By (2.5) and (2.8) and by the same observations as for the first inequality we get Lemma 2.2 $\square$

Consider meromorphic functions $F_0, \dots, F_n$ on $\mathbb{C}^m$, and put $F = (F_0, \dots, F_n)$. For each $a \in \mathbb{C}^m$, we denote by $\mathcal{M}_a$ the field of all germs of meromorphic functions on $\mathbb{C}^m$ at a and, for $p = 1, 2, \dots$ by $\mathcal{F}^p$ the $\mathcal{M}_a$-sub vector space of $\mathcal{M}_a^{n+1}$ which is generated by the set $\{D^\alpha F := (D^\alpha F_0, \dots, D^\alpha F_n) : |\alpha| \leqslant p\}$. Set $\ell_F(p) = \dim_{\mathcal{M}_a} \mathcal{F}^p$, which does not depend on $a \in \mathbb{C}^m$. As a general reference for this construction and for the following definition, see [9] and [10].

Definition 2.3. (see [10], Definition 2.10) Assume that meromorphic functions $F_0, \dots, F_n$ on $\mathbb{C}^m$ are linearly independent over $\mathbb{C}$. For $(n+1)$ vectors $\alpha^i = (\alpha_{i1}, \dots, \alpha_{im})$ $(0 \leqslant i \leqslant n)$ composed of nonnegative integers α_{ij}, we call a set $\alpha = (\alpha^0, \dots, \alpha^n)$ an admissible set for $F := (F_0, \dots, F_n)$ if

$$\left\{D^{\alpha^0} F, \dots, D^{\alpha^{\ell_F(p)-1}} F\right\}$$

is a basis of $\mathcal{F}^p$ for each $p = 1, 2, \dots, p_0 := \min\{p' : \ell_F(p') = n+1\}$.

By definition, for an admissible set $\alpha = (\alpha^0, \dots, \alpha^n)$ for $F = (F_0, \dots, F_n)$ we have

$$W^\alpha(F_0, \dots, F_n) := \det\left(D^{\alpha^0} F, \dots, D^{\alpha^n} F\right) \not\equiv 0.$$

Lemma 2.4. ([10], Proposition 2.11) *For arbitrarily given linearly independent meromorphic functions $F_0, \dots, F_n$ on $\mathbb{C}^m$, there exists an admissible set $\alpha = (\alpha^0, \dots, \alpha^n)$ with $|\alpha| := \sum_{i=0}^n |\alpha^i| \leqslant \dfrac{n(n+1)}{2}$.*

Lemma 2.5. *For arbitrarily given linearly independent meromorphic functions $F_0, \dots, F_n$ on $\mathbb{C}^m$,*

$$p_0 := \min\{p' : \ell_F(p') = n+1\} \leqslant n\,.$$

Proof. This is an easy corollary of Fujimoto [10], Proposition 2.9, since F is at least of rank one, or of Fujimoto [9], Proposition 4.5. □

Lemma 2.6. (generalization of [10], Proposition 2.12) *Let $\alpha = (\alpha^0, \dots, \alpha^n)$ be an admissible set for $F = (F_0, \dots, F_n)$ and let h be a nonzero meromorphic function on $\mathbb{C}^m$. Then*

$$W^\alpha(hF_0, \dots, hF_n) = h^{n+1} W^\alpha(F_0, \dots, F_n).$$

Proof. For holomorphic functions h this is Proposition 2.11 in [10], and its proof argument still holds for holomorphic functions defined only on a Zariski open subset of $\mathbb{C}^m$. Hence, the case of a meromorphic h follows by the identity theorem. □

We also will need the following variant of the logarithmic derivative lemma:

Lemma 2.7. *Let f be a linearly nondegenerate meromorphic map of $\mathbb{C}^m$ into $\mathbb{C}P^n$ with reduced representation $f = (f_0, \dots, f_n)$. Let $\alpha = (\alpha^0, \dots, \alpha^n)$ be an admissible set for $(f_0, \dots, f_n)$. Then*

$$\Big\| \int\limits_{S(r)} \log^+ \Big| \frac{W^\alpha(f_0, \dots, f_n)}{f_0 \cdots f_n} \Big| \sigma = o(T_f(r)).$$

Proof. By Lemma 2.6 we have

$$\int\limits_{S(r)} \log^+\Big|\frac{W^\alpha(f_0,\dots,f_n)}{f_0\cdots f_n}\Big|\sigma = \int\limits_{S(r)} \log^+\Bigg|\frac{W^\alpha\Big(1,\dfrac{f_1}{f_0},\dots,\dfrac{f_n}{f_0}\Big)}{1\cdot\dfrac{f_1}{f_0}\cdots\dfrac{f_n}{f_0}}\Bigg|\sigma$$

$$\leqslant \int\limits_{S(r)} \Bigg(K_1 \sum_{\substack{0\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} \log^+\Bigg|\frac{D^{\alpha^i}\Big(\dfrac{f_j}{f_0}\Big)}{\dfrac{f_j}{f_0}}\Bigg| + K_2\Bigg)\sigma$$

$$\leqslant K_1 \sum_{\substack{0\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} \int\limits_{S(r)} \log^+\Bigg|\frac{D^{\alpha^i}\Big(\dfrac{f_j}{f_0}\Big)}{\dfrac{f_j}{f_0}}\Bigg|\sigma + K_3\,,$$

where K_1, K_2, K_3 are constant not depending on r. On the other hand, by Theorem 2.6 in [10], we have

$$\Big\|\int\limits_{S(r)} \log^+\Bigg|\frac{D^{\alpha^i}\Big(\dfrac{f_j}{f_0}\Big)}{\Big(\dfrac{f_j}{f_0}\Big)}\Bigg| = o(T_f(r)), \quad 0\leqslant i\leqslant n, 1\leqslant j\leqslant n.$$

Hence, we get

$$\Big\|\int\limits_{S(r)} \log^+\Big|\frac{W^\alpha(f_0,\dots,f_n)}{f_0\cdots f_n}\Big|\sigma = o(T_f(r)).$$

□

We finally will need the following estimates of the divisors of such logarithmic expressions:

Proposition 2.8. (Special case of [9], Proposition 4.10) *Let f be a linearly nondegenerate meromorphic map of $\mathbb{C}^m$ into $\mathbb{CP}^n$ with reduced representation $f = (f_0 : \cdots : f_n)$. Assume that $\alpha = (\alpha^0,\dots,\alpha^n)$ is an admissible set for $F = (f_0,\dots,f_n)$, and let again $p_0 = \min\{p' : \ell_F(p') = n+1\}$. Then we have*

$$\nu_{\frac{f_0\cdot\dots\cdot f_n}{W^\alpha(f_0,\dots,f_n)}} \leqslant \sum_{i=0}^{n} \min\{\nu_{f_i}, p_0\}$$

outside an analytic set of codimension at least two.

3 Regular sequences

Throughout of this paper, we use the lexicographic order on $\mathbb{N}_0^p$. Namely, $(i_1, \ldots, i_p) > (j_1, \ldots, j_p)$ iff for some $s \in \{1, \ldots, p\}$ we have $i_\ell = j_\ell$ for $\ell < s$ and $i_s > j_s$.

Lemma 3.1. *Let A be a commutative ring and let $\{\phi_1, \ldots, \phi_p\}$ be a regular sequence in A, i.e. for $i = 1, ..., p$, ϕ_i is not a zero divisor of $A/(\phi_1, ..., \phi_{i-1})$. Denote by I the ideal in A generated by $\phi_1, \ldots, \phi_p$. Suppose that for some $q, q_1, \ldots, q_h \in A$ we have an equation*

$$\phi_1^{i_1} \cdots \phi_p^{i_p} \cdot q = \sum_{r=1}^{h} \phi_1^{j_1(r)} \cdots \phi_p^{j_p(r)} \cdot q_r \ ,$$

where $(j_1(r), \ldots, j_p(r)) > (i_1, \ldots, i_p)$ for $r = 1, \ldots, h$. Then $q \in I$.

For the proof, we refer to [6], Lemma 2.2. □

Proposition 3.2. *Let $\{Q_j\}_{j=1}^q$ $(q \geq n+1)$ be an admissible set of homogeneous polynomials of common degree $d \geq 1$ in $\mathcal{K}_f[x_0, \ldots, x_n]$. Then for any pairwise different $1 \leqslant j_0, \ldots, j_n \leqslant q$ the sequence $\{Q_{j_0}, Q_{j_1}, ..., Q_{j_n}\}$ of elements in $\mathcal{K}_{\{Q_j\}_{j=1}^q}[x_0, \ldots, x_n]$ is a regular sequence, as well as all its subsequences.*

Proof. Since $\mathcal{K}_{\{Q_j\}_{j=1}^q}$ is a field, the ring $\mathcal{K}_{\{Q_j\}_{j=1}^q}[x_0, \ldots, x_n]$ is a local Cohen-Macaulay ring with maximal ideal $\mathcal{M} = (x_0, ..., x_n) \subset \mathcal{K}_{\{Q_j\}_{j=1}^q}[x_0, \ldots, x_n]$ (see for example [15], page 112). Suppose that $\{Q_{j_0}, Q_{j_1}, ..., Q_{j_n}\}$ is a system of parameters of the ring $\mathcal{K}_{\{Q_j\}_{j=1}^q}[x_0, \ldots, x_n]$, this means (see [15], pages 73 and 78) that there exists a natural number $\rho \in \mathbb{N}$ such that

$$\mathcal{M}^\rho \subset (Q_{j_0}, Q_{j_1}, ..., Q_{j_n}) \subset \mathcal{M} \ . \tag{3.1}$$

Then by Theorem 31 of [15], any subsequence of $\{Q_{j_0}, Q_{j_1}, ..., Q_{j_n}\}$ is a regular sequence in $\mathcal{K}_{\{Q_j\}_{j=1}^q}[x_0, \ldots, x_n]$.

Since the $\{Q_j\}_{j=1}^q$ $(q \geq n+1)$ are homogeneous polynomials of common degree $d \geq 1$, the second inclusion of equation (3.1) is trivial. In order to prove the first inclusion, again by Proposition 2.1 and its notations there exists a positive integer s and polynomials $\{\widetilde{b}_{ik}\}_{0\leqslant i,k\leqslant n}$ in $\mathbb{Z}[T,x]$, zero or homogenous in x of degree $s-d$, such that

$$x_i^s\cdot\widetilde{R}=\sum_{k=0}^{n}\widetilde{b}_{ij_k}\widetilde{Q}_{j_k}\quad\text{for all } i\in\{0,\ldots,n\},$$

and, since $\{Q_{j_k}\}_{k=0}^{n}$ is an admissible set, $R=\widetilde{R}(\ldots,a_{j_kI},\ldots)\not\equiv 0$. Set

$$b_{ij_k}=\widetilde{b}_{ij_k}\big((\ldots,a_{j_kI},\ldots),(x_0,\ldots,x_n)\big),\quad 0\leqslant i,k\leqslant n.$$

Then it is clear that $R\in\mathcal{K}_{\{Q_j\}_{j=1}^q}$, $b_{ij_k}\in\mathcal{K}_{\{Q_j\}_{j=1}^q}[x_0,\ldots,x_n]$. So we get that

$$x_i^s\cdot R=\sum_{k=0}^{n}b_{ij_k}Q_{j_k}\quad\text{for all } i\in\{0,\ldots,n\},$$

implying that $x_i^s\in(Q_{j_0},Q_{j_1},...,Q_{j_n})$ for all $i=0,...,n$. So if take any $\rho\geq(n+1)(s-1)+1$, then we get the first inclusion of equation (3.1), and we are done. □

Let f be a nonconstant meromorphic map of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $\{Q_j\}_{j=1}^q$ $(q\geq n+1)$ be an admissible set of homogeneous polynomials of degree d in $\mathcal{K}_f[x_0,\ldots,x_n]$. For a nonnegative integer N, we denote by V_N the vector space (over $\mathcal{K}_{\{Q_j\}_{j=1}^q}$) consisting of all homogeneous polynomials of degree N in $\mathcal{K}_{\{Q_j\}_{j=1}^q}[x_0,\ldots,x_n]$ (and of the zero polynomial). Denote by $(Q_1,\ldots,Q_n)$ the ideal in $\mathcal{K}_{\{Q_j\}_{j=1}^q}[x_0,\ldots,x_n]$ generated by $Q_1,\ldots,Q_n$.

The following result is similar to Lemma 5 of An-Wang [2]. However, they proved it for the function field of a smooth projective variety instead of $\mathcal{K}_f$, only for sufficiently big N, and with a less elementary method, so we do not try to adopt their proof, but give an independant one.

Proposition 3.3. *Let $\{Q_j\}_{j=1}^q$ $(q\geq n+1)$ be an admissible set of homogeneous polynomials of common degree $d\geq 1$ in $\mathcal{K}_f[x_0,\ldots,x_n]$. Then for any nonnegative integer N and for any $J:=\{j_1,\ldots,j_n\}\subset\{1,\ldots,q\}$, the dimension of the vector space $\frac{V_N}{(Q_{j_1},\ldots,Q_{j_n})\cap V_N}$ is equal to the number of n-tuples $(i_1,\ldots,i_n)\in\mathbb{N}_0^n$ such that $i_1+\cdots+i_n\leqslant N$ and $0\leqslant i_1,\ldots,i_n\leqslant d-1$. In particular, for all $N\geq n(d-1)$, we have*

$$\dim\frac{V_N}{(Q_{j_1},\ldots,Q_{j_n})\cap V_N}=d^n.$$

Proof. The case $N = 0$ holds trivially, so we assume that N is positive for the rest of the proof. We first prove that

$$\dim\frac{V_N}{(Q_{j_1},\dots,Q_{j_n})\cap V_N} = \dim\frac{V_N}{(Q_1,\dots,Q_n)\cap V_N} \tag{3.2}$$

for any choice of $J := \{j_1,\dots,j_n\} \in \{1,\dots,q\}$ and any N. For this it suffices to prove that

$$\dim(Q_1,\dots,Q_n)\cap V_N = \dim(Q_{j_1},\dots,Q_{j_n})\cap V_N\ .$$

Since the order of the Q_j does not matter, it suffices to prove

$$\dim(Q_1,\dots,Q_n)\cap V_N = \dim(Q_1,\dots,Q_{n-1},Q_{j_n})\cap V_N\ , \tag{3.3}$$

the rest follows by induction. But for (3.3) it suffices to prove:

$$\dim\frac{(Q_1,\dots,Q_n)\cap V_N}{(Q_1,\dots,Q_{n-1})\cap V_N} = \dim\frac{(Q_1,\dots,Q_{n-1},Q_{j_n})\cap V_N}{(Q_1,\dots,Q_{n-1})\cap V_N}\ . \tag{3.4}$$

We denote for simplicity $\mathcal{K} := \mathcal{K}_{\{Q_j\}_{j=1}^q}$ and let ϕ be the following $\mathcal{K}$-linear map:

$$\phi : \frac{(Q_1,\dots,Q_n)\cap V_N}{(Q_1,\dots,Q_{n-1})\cap V_N} \to \frac{(Q_1,\dots,Q_{n-1},Q_{j_n})\cap V_N}{(Q_1,\dots,Q_{n-1})\cap V_N}\ ;$$

$$[\sum_{j=1}^{n-1} b_jQ_j + b_nQ_n] \mapsto [\sum_{j=1}^{n-1} b_jQ_j + b_nQ_{j_n}]$$

with $b_j \in \mathcal{K}[x_0,\dots,x_n]$. This map is clearly surjective, so if we still prove that it is well defined and injective, we get (3.4). In order to prove that ϕ is well defined, let $[\sum_{j=1}^{n-1} b_jQ_j + b_nQ_n] = [\sum_{j=1}^{n-1} b'_jQ_j + b'_nQ_n]$. This means that $(b_n - b'_n)Q_n \in (Q_1,\dots,Q_{n-1})\cap V_N$. But since by Proposition 3.2, $Q_1,\dots,Q_n$ is a regular sequence, Q_n is not a zero divisor in $\frac{\mathcal{K}[x_0,\dots,x_n]}{(Q_1,\dots,Q_{n-1})}$, so that $(b_n - b'_n) \in (Q_1,\dots,Q_{n-1})$. Hence,

$$[\sum_{j=1}^{n-1} b_jQ_j + b_nQ_{j_n}] - [\sum_{j=1}^{n-1} b'_jQ_j + b'_nQ_{j_n}] = [(b_n - b'_n)Q_{j_n}] = 0$$

in $\frac{(Q_1,\dots,Q_{n-1},Q_{j_n})\cap V_N}{(Q_1,\dots,Q_{n-1})\cap V_N}$, so ϕ is well defined. The injectivity of ϕ follows by the same argument, just changing the roles of Q_n and Q_{j_n}, since by Proposition 3.2, $Q_1,\dots,Q_{n-1},Q_{j_n}$ is also a regular sequence. Hence, we get (3.3) and, thus, (3.2). We finally remark that for the proof of (3.3) we only used that $\{Q_1, ...Q_n, Q_{j_n}\}$ is an admissible set of homogenous polynomials of common degree d.

Take a point $z_0 \in \mathbb{C}^m$ such that the hypersufaces in $\mathbb{C}P^n$ defined by $Q_1(z_0), \ldots, Q_{n+1}(z_0)$ have no common point. Since $Q_1(z_0), \ldots, Q_n(z_0)$ define a subvariety of dimension 0, there exists a hyperplane H_{n+1} in $\mathbb{C}P^n$ such that $\cap_{i=1}^{n} Q_j(z_0) \cap H_{n+1} = \varnothing$. Furthermore, by induction, there exist hyperplanes $H_1, \ldots, H_{n+1}$ such that $\cap_{j=1}^{i-1} Q_j(z_0) \cap_{k=i}^{n+1} H_k = \varnothing$, for all $i \in \{1, \ldots, n+1\}$. This means that $\{Q_1, \ldots, Q_{i-1}, H_i^d, \ldots, H_{n+1}^d\}$ is an admissible set, for any $i \in \{1, \ldots, n+1\}$. Then, by (3.3), taking into accont the remark at the end of its proof, and by induction, we get that

$$\dim \frac{V_N}{(Q_1, \ldots, Q_n) \cap V_N} = \dim \frac{V_N}{(H_1^d, \ldots, H_n^d) \cap V_N}. \tag{3.5}$$

As $H_1, \ldots, H_{n+1}$ are linearly independent, it follows from a well-known fact of linear algebra that there exists a permutation $\{k_1, \ldots, k_{n+1}\}$ of $\{0, \ldots, n\}$ such that $H_1, \ldots, H_{i-1}, x_{k_i}, \ldots, x_{k_{n+1}}$ are linearly independent, for any $i \in \{1, \ldots, n+2\}$. This means that $\{H_1^d, \ldots, H_{i-1}^d, x_{k_i}^d, \ldots, x_{k_{n+1}}^d\}$ is an admissible set. Then, by (3.3) and by induction, we get that

$$\dim \frac{V_N}{(H_1^d, \ldots, H_n^d) \cap V_N} = \dim \frac{V_N}{(x_1^d, \ldots, x_n^d) \cap V_N}. \tag{3.6}$$

By (3.2), (3.5) and (3.6), for all positive integer N we have

$$\dim \frac{V_N}{(Q_{j_1}, \ldots, Q_{j_n}) \cap V_N} = \dim \frac{V_N}{(x_1^d, \ldots, x_n^d) \cap V_N}.$$

On the other hand, it is easy to see that for any positive integer N, the vector space $\frac{V_N}{(x_1^d, \ldots, x_n^d) \cap V_N}$ has a basis $\{[x_0^{N-(i_1+\cdots i_n)} x_1^{i_1} \cdots x_n^{i_n}], i_1 + \cdots + i_n \leqslant N, 0 \leqslant i_1, \ldots, i_n \leqslant d-1\}$. This completes the proof of Proposition 3.3. $\square$

4 Proof of Main Theorem

We first prove the theorem for the case where all the Q_j $(j = 1, \ldots, q)$ have the same degree d.

We may assume, without loss of generality, that f is algebraically nondegenerate over $\mathcal{K}_{\{Q_j\}_{j=1}^q}$: We replace the polynomials $\{Q_j\}_{j=1}^q$ by the polynomials $\tilde{Q}_j := \frac{1}{a_{jI_2}} Q_j$, where $a_{jI_2} \neq 0$ is any nonzero coefficient of Q_j,

$j = 1, ..., q$. Then $\{\tilde{Q}_j\}_{j=1}^q$ is also an admissible set of homogeneous polynomials in $\mathcal{K}_f[x_0, \ldots, x_n]$ with $\deg Q_j = d \geq 1$. Since f is algebraically nondegenerate over $\tilde{\mathcal{K}}_{\{Q_j\}_{j=1}^q}$ and $\tilde{\mathcal{K}}_{\{Q_j\}_{j=1}^q} \supset \mathcal{K}_{\{\tilde{Q}_j\}_{j=1}^q}$, we have that f is algebraically nondegenerate over $\mathcal{K}_{\{\tilde{Q}_j\}_{j=1}^q}$. So we get that for any $\varepsilon > 0$, there exists a positive integer L, depending on n, ϵ and d in an explicit way, such that

$$\|(q-n-1-\varepsilon)T_f(r) \leqslant \sum_{j=1}^{q} \frac{1}{d} N_f^{(L)}(r, \tilde{Q}_j).$$

But since

$$\begin{aligned} N_f^{(L)}(r, \tilde{Q}_j) &= N_f^{(L)}(r, \frac{1}{a_{jI_2}} Q_j) = N^{(L)}_{\frac{1}{a_{jI_2}} Q_j \circ f}(r) \\ &\leqslant N_{\frac{1}{a_{jI_2}}}(r) + N^{(L)}_{Q_j \circ f}(r) = N^{(L)}(r, Q_j \circ f) + o(T_f(r)), \end{aligned}$$

we finally get that for any $\varepsilon > 0$ we have

$$\|(q-n-1-\varepsilon)T_f(r) \leqslant \sum_{j=1}^{q} \frac{1}{d} N_f^{(L)}(r, Q_j).$$

For each nonnegative integer k, we denote again by V_k the space (over $\mathcal{K}_{\{Q_j\}_{j=1}^q}$) of homogeneous polynomials of degree k (and of the zero polynomial) in $\mathcal{K}_{\{Q_j\}_{j=1}^q}[x_0, \ldots, x_n]$. Set $V_k = \{0\}$ for $k < 0$.

Let $J := \{j_1, \ldots, j_n\} \subset \{1, \ldots, q\}$. For each positive integer N divisible by d and for each $I := (i_1, \ldots, i_n) \in \mathbb{N}_0^n$ with $\|I\| := \sum_{s=1}^{n} i_s \leqslant \frac{N}{d}$, we set

$$V_N^I = \sum_{E:=(e_1,\ldots,e_n) \geq I} Q_{j_1}^{e_1} \cdots Q_{j_n}^{e_n} \cdot V_{N-d\|E\|}.$$

Note that $V_N^I \supset V_N^J$ if $I < J$ (lexicographic order), and $V_N^{(0,\ldots,0)} = V_N$.

Denote by $\{I_1, \ldots, I_K\}$ the set of all $I \in \mathbb{N}_0^n$ with $\|I\| \leqslant \frac{N}{d}$. We write $I_k = (i_{1k}, \ldots, i_{nk})$, $k = 1, \ldots, K$. Assume that $I_1 = (0, \ldots, 0) < I_2 < \cdots <$

$I_K = (\frac{N}{d}, 0, ..., 0)$. We have

$$V_N = V_N^{I_1} \supset V_N^{I_2} \supset \cdots \supset V_N^{I_K} = \{u \cdot Q_{j_1}^{\frac{N}{d}} : u \in \mathcal{K}_{\{Q_j\}_{j=1}^q}\}$$
$$\text{and } K = K(N, d, n) = \binom{\frac{N}{d} + n}{n}.$$

Set

$$m_k := \dim \frac{V_N^{I_k}}{V_N^{I_{k+1}}}, \quad k = 1, \ldots, K-1, \text{ and } m_K := \dim V_N^{I_K} = 1.$$

We now prove that:
Although the $V_N^{I_k}$ may depend on J, the m_k, $k = 1, \ldots, K$ are independent of J. Moreover,

$$m_k = d^n \tag{4.1}$$

for all N divisible by d and for all $k \in \{1, \ldots, K\}$ with $N - d\|I_k\| \geq nd$.

We define vector space homomorphisms

$$\varphi_k : V_{N-d\|I_k\|} \longrightarrow \frac{V_N^{I_k}}{V_N^{I_{k+1}}} \quad (k = 1, \ldots, K-1)$$

as $\varphi_k(\gamma) = \left[Q_{j_1}^{i_1 k} \cdots Q_{j_n}^{i_n k}\gamma\right]$, where $\gamma \in V_{N-d\|I_k\|}$ and $\left[Q_{j_1}^{i_1 k} \cdots Q_{j_n}^{i_n k}\gamma\right]$ is the class in $\frac{V_N^{I_k}}{V_N^{I_{k+1}}}$ containing $Q_{j_1}^{i_1 k} \cdots Q_{j_n}^{i_n k}\gamma$.

It is clear that the φ_k $(1 \leqslant k \leqslant K-1)$ are surjective (note that for any $E \in \mathbb{N}_0^n$ with $\|E\| \leqslant \dfrac{N}{d}$ and $E > I_k$ then $E \geq I_{k+1}$).

For any $\gamma \in \ker \varphi_k$

$$\begin{aligned} Q_{j_1}^{i_1 k} \cdots Q_{j_n}^{i_n k}\gamma &\in \sum_{E=(e_1,\ldots,e_n)\geq I_{k+1}} Q_{j_1}^{e_1} \cdots Q_{j_n}^{e_n} V_{N-d\|E\|} \\ &= \sum_{E=(e_1,\ldots,e_n) > I_k} Q_{j_1}^{e_1} \cdots Q_{j_n}^{e_n} V_{N-d\|E\|}. \end{aligned}$$

So we have

$$Q_{j_1}^{i_1 k} \cdots Q_{j_n}^{i_n k}\gamma = \sum_{E=(e_1,\ldots,e_n) > I_k} Q_{j_1}^{e_1} \cdots Q_{j_n}^{e_n} \gamma_E \ ,$$

where $\gamma_E \in V_{N-d\|E\|}$. Furthermore, by Lemma 3.1 and Proposition 3.2 we have $\gamma \in (Q_{j_1}, \ldots, Q_{j_n})$. Thus

$$\ker \varphi_k \subset (Q_{j_1}, \ldots, Q_{j_n}) \cap V_{N-d\|I_k\|} \, . \tag{4.2}$$

Conversely, for any $\gamma \in (Q_{j_1}, \ldots, Q_{j_n}) \cap V_{N-d\|I_k\|}$ $(\gamma \neq 0)$,

$$\gamma = \sum_{s=1}^{n} \gamma_s Q_{j_s}, \quad \gamma_s \in V_{N-d(\|I_k\|+1)},$$

we have

$$I'_s := (i_{1k}, \ldots, i_{sk} + 1, \ldots, i_{nk}) > I_k, \quad (s = 1, \ldots, n)$$

and $\|I'_s\| = \|I_k\| + 1 \leqslant \dfrac{N}{d}$ (since $\gamma \neq 0$). So we get $I'_s \geq I_{k+1}$, $s = 1, \ldots, n$. Thus

$$Q_{j_1}^{i_{1k}} \cdots Q_{j_n}^{i_{nk}} \gamma = \sum_{s=1}^{n} Q_{j_1}^{i_{1k}} \cdots Q_{j_s}^{i_{sk}+1} \cdots Q_{j_n}^{i_{nk}} \cdot \gamma_s \in V_N^{I_{k+1}} \, .$$

This means that $\gamma \in \ker \varphi_k$. So we have

$$\ker \varphi_k \supset (Q_{j_1}, \ldots, Q_{j_n}) \cap V_{N-d\|I_k\|} \, . \tag{4.3}$$

By (4.2), (4.3) and since φ_k is surjective, we have:

$$m_k = \dim \frac{V_N^{I_k}}{V_N^{I_{k+1}}} \simeq \dim \frac{V_{N-d\|I_k\|}}{(Q_{j_1}, \ldots, Q_{j_n}) \cap V_{N-d\|I_k\|}} \, , \; k \in \{1, ..., K-1\}. \tag{4.4}$$

Hence, by Proposition 3.3 we get (4.1) and the independence of J of m_k. $\square$

Since $V_N = V_N^{I_1} \supset V_N^{I_2} \supset \cdots \supset V_N^{I_K}$ and $m_k = \dim \dfrac{V_N^{I_k}}{V_N^{I_{k+1}}}$, $(k \in \{1, ..., K-1\})$, $m_K = \dim V_N^{I_K} = 1$, we may choose a basis $\{\psi_1^J, \ldots, \psi_M^J\}$ $(M = \binom{N+n}{n})$ of V_N such that

$$\left\{\psi^J_{M-(m_k+\cdots+m_K)+1}, \ldots, \psi^J_M\right\}$$

is a basis of $V_N^{I_k}$ for any $k \in \{1,\dots,K\}$. For each $k \in \{1,\dots,K\}$ and $\ell \in \big\{M-(m_{k+1}+\cdots+m_K),\dots,M-(m_k+\cdots+m_K)+1\big\}$, we have

$$\psi_\ell^J = Q_{j_1}^{i_{1k}}\cdots Q_{j_n}^{i_{nk}}\gamma_\ell^J, \text{ where } \gamma_\ell^J \in V_{N-d\|I_k\|}. \tag{4.5}$$

Then, we have

$$\prod_{j=1}^{M}\psi_j^J(f) = \prod_{k=1}^{K}\Big((Q_{j_1}(f))^{i_{1k}}\cdots(Q_{j_n}(f))^{i_{nk}}\Big)^{m_k}\cdot\prod_{\ell=1}^{M}\gamma_\ell^J(f) \tag{4.6}$$

By Lemma 2.2 there exists $h_\ell^J \in \mathcal{C}_f$ such that, outside an analytic subset in $\mathbb{C}^m$ of codimension at least two,

$$|\gamma_\ell^J(f)| \leqslant h_\ell^J\cdot\|f\|^{N-d\|I_k\|}.$$

So we get

$$\prod_{\ell=1}^{M}|\gamma_\ell^J(f)| \leqslant \prod_{k=1}^{K}\big(\|f\|^{N-d\|I_k\|}\big)^{m_k}\cdot h^J\,,$$

where $h^J := \prod_{\ell=1}^{M} h_\ell^J \in \mathcal{C}_f$. This implies that (outside a proper analytic subset of $\mathbb{C}^m$)

$$\log\prod_{\ell=1}^{M}|\gamma_\ell^J(f)| \leqslant \sum_{k=1}^{K} m_k(N-d\|I_k\|)\log\|f\| + \log h^J. \tag{4.7}$$

By (4.4) and since $m_K=1$, we have that m_k only depends on $\|I_k\|$, i.e. $m_k = m(\|I_k\|)$, $k=1,\dots,K$. So we have, for $s=1,...,n$,

$$\sum_{k=1}^{K} m_k\cdot i_{sk} = \sum_{\ell=0}^{\frac{N}{d}}\sum_{k:\|I_k\|=\ell} m_k\cdot i_{sk} = \sum_{\ell=0}^{\frac{N}{d}} m(\ell)\sum_{k:\|I_k\|=\ell} i_{sk}\,.$$

Now for every ℓ the the symmetry $(i_1,\cdots,i_n)\to((i_{\sigma(1)},\dots,i_{\sigma(n)})$ shows that $\sum_{k:\|I_k\|=\ell} i_{sk}$ is independent of s. So, we get

$$A := \sum_{k=1}^{K} m_k\cdot i_{sk} \quad \text{is independent of} \quad s \quad \text{and} \quad J\,, \tag{4.8}$$

the latter by (4.1).

Denote by $\mathcal{B}$ the set of all $k \in \{1,\ldots,K\}$ such that $N - d\|I_k\| \geq nd$. Put

$$\widetilde{I}_k := (i_{1k},\ldots,i_{nk},i_{(n+1)k}), \quad k \in \mathcal{B},$$

where $i_{(n+1)k} := (\frac{N}{d} - n) - (i_{1k} + \cdots + i_{nk})$. Then $\{\widetilde{I}_k : \ k \in \mathcal{B}\}$ is the set of all $\widetilde{I} \in \mathbb{N}_0^{n+1}$ with $\|\widetilde{I}\| = \frac{N}{d} - n$. For any $\widetilde{I} := (i_1,\ldots,i_{n+1}) \in \{\widetilde{I}_k : k \in \mathcal{B}\}$ and for any bijection $\sigma : \{1,\ldots,n+1\} \to \{1,\ldots,n+1\}$, we have $(i_{\sigma(1)},\ldots,i_{\sigma(n+1)}) \in \{\widetilde{I}_k : \ k \in \mathcal{B}\}$. Therefore, by (4.1) we have

$$A \geq \sum_{k\in\mathcal{B}} m_k \cdot i_{sk} = d^n \sum_{k\in\mathcal{B}} i_{sk} = \frac{d^n}{n+1}\sum_{k\in\mathcal{B}} \|\widetilde{I}_k\| = \frac{d^n}{n+1}\binom{\frac{N}{d}}{n}(\frac{N}{d} - n). \tag{4.9}$$

We have

$$\sum_{k=1}^{K} m_k \cdot (\frac{N}{d} - \|I_k\|) = \sum_{k=1}^{K} m_k \cdot \frac{N}{d} - \sum_{k=1}^{K} m_k \cdot \|I_k\| = \frac{MN}{d} - nA. \tag{4.10}$$

By (4.6), we have for N divisible by d:

$$\prod_{j=1}^{M} \psi_j^J(f) = \big(Q_{j_1}(f)\cdots Q_{j_n}(f)\big)^A \cdot \prod_{l=1}^{M} \gamma_\ell^J(f) \tag{4.11}$$

By (4.7), (4.10) and (4.11), we have (outside a proper analytic subset of $\mathbb{C}^m$) for N divisible by d :

$$\log\prod_{j=1}^{M} |\psi_j^J(f)| \leqslant A \cdot \log\prod_{i=1}^{n} |Q_{j_i}(f)| + (\frac{MN}{d} - nA)d \cdot \log\|f\| + \log h^J\,.$$

If we still choose the function $h \in \mathcal{C}_f$, with $h \geq 1$, common for all J, for example by putting $h := \prod_J (1 + h_J))$, we get

$$\log\prod_{i=1}^{n} |Q_{j_i}(f)| \geq \frac{1}{A}(\log\prod_{j=1}^{M} |\psi_j^J(f)| - \log h) - (\frac{MN}{dA} - n)d \cdot \log\|f\|. \tag{4.12}$$

We choose $N := d\cdot[2(n+1)(2^n-1)(nd+1)\epsilon^{-1}+n+1]$. Then by (4.9), we have (assuming without loss of generality that $\epsilon<1$)

$$d\cdot(\frac{MN}{dA}-n-1)\leqslant d\cdot\Big(\frac{N\binom{N+n}{n}}{\frac{d^{n+1}}{n+1}\binom{\frac{N}{d}}{n}(\frac{N}{d}-n)}-n-1\Big)$$
$$=d(n+1)\cdot\Big(\prod_{i=1}^{n}\frac{N+i}{N-(n+1-i)d}-1\Big)<d(n+1)\Big((\frac{N+1}{N-nd})^n-1\Big)$$
$$=d(n+1)\Big((1+\frac{nd+1}{N-nd})^n-1\Big)<d(n+1)(2^n-1)\frac{nd+1}{N-nd}$$
$$\leqslant d(n+1)(2^n-1)\frac{nd+1}{d\cdot\big(2(n+1)(2^n-1)(nd+1)\epsilon^{-1}+n\big)-nd}=\frac{\epsilon}{2}. \tag{4.13}$$

By (4.12) and Lemma 2.2 (applied to every factor Q_{β_j}, $j=1,\dots,q-n$, using that we can complete every Q_{β_j} with n other Q_j not having bigger norm, so that the maximum of the norms is obtained by Q_{β_j}), we have

$$\log\prod_{j=1}^{q}|Q_j(f)|=\max_{\{\beta_1,\dots,\beta_{q-n}\}\subset\{1,\dots,q\}}\log|Q_{\beta_1}(f)\cdots Q_{\beta_{q-n}}(f)|$$
$$+\min_{J=\{j_1,\dots,j_n\}\subset\{1,\dots,q\}}\log|Q_{j_1}(f)\cdots Q_{j_n}(f)|$$
$$\geq(q-n)d\cdot\log\|f\|+\min_{J\subset\{1,\dots,q\}}\frac{1}{A}\log\prod_{j=1}^{M}|\psi_j^J(f)|$$
$$-d\cdot(\frac{MN}{Ad}-n)\log\|f\|-\log\widetilde{h}$$
$$=(q-n-1)d\cdot\log\|f\|+\frac{1}{A}\min_{J\subset\{1,\dots,q\}}\log\prod_{j=1}^{M}|\psi_j^J(f)|$$
$$-d\cdot(\frac{MN}{Ad}-n-1)\cdot\log\|f\|-\log\widetilde{h},$$

where the choices of the indices for the maximum respectively the minimum may depend on z, however, by (observing $A\geq 1$ and by) choosing $\widetilde{h}$ as a product of the form $\prod(1+h_\nu)$, where the h_ν run over all the possible choices, we obtain $\widetilde{h} \in \mathcal{C}_f$. Furthermore we observe that the first and the last term are well defined outside an analytic subset of $\mathbb{C}^m$ of codimension at least two and the choices of maxima and minima are locally finite there, in particular the resulting functions are continuous there as functions with values in $[0, +\infty]$. Hence, the inequality still holds outside an analytic subset of $\mathbb{C}^m$ of codimension at least two by continuity. So by integrating and by using (4.13), outside an analytic subset of codimension at least two in $\mathbb{C}^m \supset S(r)$ we get

$$\int\limits_{S(r)} \log \prod_{j=1}^{q} |Q_j(f)|\sigma \geq (q-n-1)d \cdot T_f(r) + \frac{1}{A} \int\limits_{S(r)} \min_J \log \prod_{j=1}^{M} |\psi_j^J(f)|\sigma - \frac{\varepsilon}{2} T_f(r) - o(T_f(r)). \qquad (4.14)$$

We write

$$\psi_j^J = \sum_{I \in \mathcal{T}_N} c_{jI}^J x^I \in V_N, \quad c_{jI}^J \in \mathcal{K}_{\{Q_j\}_{j=1}^q}$$

with $j = 1, \ldots, M,\ J \subset \{1, \ldots, q\},\ \#J = n$. For each $j \in \{1, \ldots, M\}$ and $J \subset \{1, \ldots, q\}$, $\#J = n$ we fix an index $I_i^J \in \mathcal{T}_N$ such that $c_{jI_j^J}^J \not\equiv 0$. Define

$$\xi_{jI}^J = \frac{c_{jI}^J}{c_{jI_j^J}^J}, \quad j \in \{1, \ldots, M\},\ J \subset \{1, \ldots, q\},\ \#J = n.$$

Set

$$\widetilde{\psi}_j^J = \sum_{I \in \mathcal{T}_N} \xi_{jI}^J x^I \in \mathcal{K}_{\{Q_j\}_{j=1}^q}[x_0, \ldots, x_n].$$

For each positive integer p, we denote by $\mathcal{L}(p)$ the vector space generated over $\mathbb{C}$ by

$$\Big\{ \prod_{\substack{1 \leqslant j \leqslant M, I \in \mathcal{T}_N \\ J \subset \{1,\ldots,q\}, \#J = n}} \big(\xi_{jI}^J\big)^{n_{jI}^J} : n_{jI}^J \in \mathbb{N}_0, \sum_{\substack{1 \leqslant j \leqslant M, I \in \mathcal{T}_N \\ J \subset \{1,\ldots,q\}, \#J = n}} n_{jI}^J = p \Big\}.$$

We have $\mathcal{L}(p) \subset \mathcal{L}(p+1) \subset \mathcal{K}_{\{Q_j\}_{j=1}^q}$ (note that $\xi_{jI_j^J}^J \equiv 1$, $j \in \{1, \ldots, M\}$, $J \subset \{1, \ldots, q\}$, $\#J = n$). Let $\{b_1, \ldots, b_{t_{p+1}}\}$ be a basis of $\mathcal{L}(p+1)$ such that

$\{b_1, \ldots, b_{t_p}\}$ is a basis of $\mathcal{L}(p)$. It is easy to see that

$$t_{p+1} \leqslant \begin{pmatrix} \binom{n+N}{n}^2 . \binom{q}{n} + p \\ \binom{n+N}{n}^2 . \binom{q}{n} - 1 \end{pmatrix} < \left(\binom{n+N}{n}^2 . \binom{q}{n} + p \right)^{\binom{n+N}{n}^2 . \binom{q}{n} - 1} \tag{4.15}$$

for all positive integer p (note that $\#\mathcal{T}_N = \binom{n+N}{n} = M$).

Since $\{\widetilde{\psi}_1^J, \ldots, \widetilde{\psi}_M^J\}$ is a basis of V_N, $\{b_1, \ldots, b_{t_{p+1}}\}$ is a basis of $\mathcal{L}(p+1)$ and f is algebraically nondegenerate over $\mathcal{K}_{\{Q_j\}_{j=1}^q}$ we have that $b_k \widetilde{\psi}_j^J(f)$ $(1 \leqslant k \leqslant t_{p+1},\ 1 \leqslant j \leqslant M)$ are linearly independent over $\mathbb{C}$. It is easy to see that $b_\ell \widetilde{\psi}_j^J(f)$ $(1 \leqslant \ell \leqslant t_p,\ 1 \leqslant j \leqslant M,\ J \subset \{1, \ldots, q\},\ \#J = n)$ are linear combinations of $b_k f^I$ $(1 \leqslant k \leqslant t_{p+1},\ I \in \mathcal{T}_N)$ over $\mathbb{C}$. So for each $J \subset \{1, \ldots, q\}$, $\#J = n$ there exists $A_J \in \mathrm{mat}(t_{p+1}M \times t_p M, \mathbb{C})$ such that

$$\left(b_\ell \widetilde{\psi}_j^J(f), 1 \leqslant \ell \leqslant t_p, 1 \leqslant j \leqslant M\right) = \left(b_k f^I, 1 \leqslant k \leqslant t_{p+1}, I \in \mathcal{T}_N\right) \cdot A_J$$

(note that $\#\mathcal{T}_N = M$). Since $b_\ell \widetilde{\psi}_j^J(f)$ $(1 \leqslant \ell \leqslant t_p,\ 1 \leqslant j \leqslant M)$ are linearly independent over $\mathbb{C}$, we obtain $\operatorname{rank} A_J = t_p M$.
Take matrices $B_J \in \mathrm{mat}(t_{p+1}M \times (t_{p+1} - t_p)M, \mathbb{C})$ $(J \subset \{1, \ldots, q\}, \#J = n)$ such that

$$C_J := A_J \cup B_J \in GL(t_{p+1}M, \mathbb{C}).$$

We write

$$\begin{aligned}\left(b_k f^I, 1 \leqslant k \leqslant t_{p+1}, I \in \mathcal{T}_N\right) \cdot C_J = \big(b_\ell \widetilde{\psi}_j^J(f), 1 \leqslant \ell \leqslant t_p,\ 1 \leqslant j \leqslant M \\ h_{uv}^J, t_p + 1 \leqslant u \leqslant t_{p+1},\ 1 \leqslant v \leqslant M\big).\end{aligned} \tag{4.16}$$

Since $\{b_1, \ldots, b_{t_{p+1}}\}$ is a basis of $\mathcal{L}(p+1)$ and f is algebraically nondegenerate over $\mathcal{K}_{\{Q_j\}_{j=1}^q}$ we have that $b_k f^I$ $(1 \leqslant k \leqslant t_{p+1}, I \in \mathcal{T}_N)$ are linearly independent over $\mathbb{C}$. By Lemma 2.4 there exists an admissible set $\alpha := (\alpha^0, \ldots, \alpha^{t_{p+1}M})$ for $(b_k f^I, 1 \leqslant k \leqslant t_{p+1}, I \in \mathcal{T}_N)$. By (4.16) we have that α is also an admissible set for

$$\left(b_\ell \widetilde{\psi}_j^J(f), 1 \leqslant \ell \leqslant t_p, 1 \leqslant j \leqslant M, h_{uv}^J, t_p + 1 \leqslant u \leqslant t_{p+1}, 1 \leqslant v \leqslant M\right).$$

Set $W^\alpha := W^\alpha(b_k f^I, 1 \leqslant k \leqslant t_{p+1}, I \in \mathcal{T}_N)$ and

$W_J^\alpha := W^\alpha(b_\ell \widetilde{\psi}_j^J(f), 1 \leqslant \ell \leqslant t_p, 1 \leqslant j \leqslant M, h_{uv}^J, t_p{+}1 \leqslant u \leqslant t_{p+1}, 1 \leqslant v \leqslant M)$.

We have $W_J^\alpha = \det C_J \cdot W^\alpha$.

By Lemma 2.2 we have (with the same arguments on domains of definition and continuity as above)

$$\int\limits_{S(r)} \min_J \log \prod_{j=1}^{M} |\psi_j^J(f)|^{t_p} \sigma \geq \int\limits_{S(r)} \min_J \log \prod_{\substack{1 \leqslant j \leqslant M \\ 1 \leqslant \ell \leqslant t_p}} |b_\ell \widetilde{\psi}_j^J(f)| \sigma$$

$$+ \int\limits_{S(r)} \min_J \log \frac{\prod_{j=1}^{M} |c^J_{jI_j^J}|^{t_p}}{\prod_{\ell=1}^{t_p} |b_\ell|^M} \sigma$$

$$\geq \int\limits_{S(r)} \min_J \log \prod_{\substack{1 \leqslant j \leqslant M \\ 1 \leqslant \ell \leqslant t_p}} |b_\ell \widetilde{\psi}_j^J(f)| \sigma - o(T_f(r))$$

$$\geq \int\limits_{S(r)} \min_J \log \Big(\prod_{\substack{1 \leqslant j \leqslant M \\ 1 \leqslant \ell \leqslant t_p}} |b_\ell \widetilde{\psi}_j^J(f)| \cdot \prod_{\substack{t_p+1 \leqslant u \leqslant t_{p+1} \\ 1 \leqslant v \leqslant M}} |h_{uv}^J| \Big) \sigma$$

$$- \int\limits_{S(r)} \max_J \log \prod_{\substack{t_p+1 \leqslant u \leqslant t_{p+1} \\ 1 \leqslant v \leqslant M}} |h_{uv}^J| \sigma - o(T_f(r))$$

$$\geq \int\limits_{S(r)} \min_J \log \Big(\prod_{\substack{1 \leqslant j \leqslant M \\ 1 \leqslant \ell \leqslant t_p}} |b_\ell \widetilde{\psi}_j^J(f)| \cdot \prod_{\substack{t_p+1 \leqslant u \leqslant t_{p+1} \\ 1 \leqslant v \leqslant M}} |h_{uv}^J| \Big) \sigma$$

$$- \int\limits_{S(r)} \log \|f\|^{NM(t_{p+1}-t_p)} \sigma - o(T_f(r))$$

$$\geq \int\limits_{S(r)} \min_J \log \Big(\prod_{\substack{1 \leqslant j \leqslant M \\ 1 \leqslant \ell \leqslant t_p}} |b_\ell \widetilde{\psi}_j^J(f)| \cdot \prod_{\substack{t_p+1 \leqslant u \leqslant t_{p+1} \\ 1 \leqslant v \leqslant M}} |h_{uv}^J| \Big) \sigma$$

$$- NM(t_{p+1} - t_p) T_f(r) - o(T_f(r)) \,, \tag{4.17}$$

where $\min_J$ is taken over all subset $J \subset \{1, \ldots, q\}$, $\#J = n$.

We may choose a positive integer p such that

$$p \leqslant p_0 := \Big[\frac{\big(\binom{n+N}{n}^2.\binom{q}{n} - 1\big).\log\big(\binom{n+N}{n}^2.\binom{q}{n}\big)}{\log(1+\frac{\epsilon}{2MN})} + 1\Big]^2 \text{ and } \frac{t_{p+1}}{t_p} < 1 + \frac{\epsilon}{2MN}. \tag{4.18}$$

Indeed, otherwise $\frac{t_{p+1}}{t_p} \geq 1 + \frac{\epsilon}{2MN}$ for all $p \leqslant p_0$. This implies that $t_{p_0+1} \geq (1+\frac{\epsilon}{2MN})^{p_0}$. Therefore, by (4.15) we have

$$\begin{aligned}
\log(1+\frac{\epsilon}{2MN}) \leqslant \frac{\log t_{p_0+1}}{p_0} &< \frac{\big(\binom{n+N}{n}^2.\binom{q}{n} - 1\big).\log\big(\binom{n+N}{n}^2.\binom{q}{n} + p_0\big)}{p_0}\\
&\leqslant \frac{\big(\binom{n+N}{n}^2.\binom{q}{n} - 1\big).\log\big(\binom{n+N}{n}^2.\binom{q}{n}\big)\cdot \log p_0}{p_0}\\
&\leqslant \frac{\big(\binom{n+N}{n}^2.\binom{q}{n} - 1\big).\log\big(\binom{n+N}{n}^2.\binom{q}{n}\big)\cdot \sqrt{p_0}}{p_0}\\
&< \log(1+\frac{\epsilon}{2MN}).
\end{aligned}$$

This is a contradiction.

We now fix a positive integer p satisfying (4.18).
By (4.17), we have

$$\begin{aligned}
&\int\limits_{S(r)} \min_J \log\prod_{j=1}^{M}|\psi_j^J(f)|\sigma \geq \frac{1}{t_p}\int\limits_{S(r)} \min_J \log\Big(\prod_{\substack{1\leqslant j\leqslant M\\ 1\leqslant \ell\leqslant t_p}} |b_\ell\widetilde{\psi}_j^J(f)| \cdot \prod_{\substack{t_p+1\leqslant u\leqslant t_{p+1}\\ 1\leqslant v\leqslant M}} |h_{uv}^J|\Big)\sigma\\
&\quad - \frac{\varepsilon}{2}T_f(r) - o(T_f(r))\\
&\geq \frac{1}{t_p}\int\limits_{S(r)} \min_J \log\frac{\prod\limits_{\substack{1\leqslant j\leqslant M\\ 1\leqslant \ell\leqslant t_p}} |b_\ell\widetilde{\psi}_j^J(f)| \cdot \prod\limits_{\substack{t_p+1\leqslant u\leqslant t_{p+1}\\ 1\leqslant v\leqslant M}} |h_{uv}^J|}{|W_J^\alpha|}\sigma\\
&\quad + \frac{1}{t_p}\int\limits_{S(r)} \min_J \log|W_J^\alpha|\sigma - \frac{\varepsilon}{2}T_f(r) - o(T_f(r))
\end{aligned}$$

$$\geq -\frac{1}{t_p}\int\limits_{S(r)} \max_J \log^+ \frac{|W_J^\alpha|}{\prod\limits_{\substack{1\leqslant j\leqslant M\\ 1\leqslant \ell\leqslant t_p}} |b_\ell \widetilde{\psi}_j^J(f)| \cdot \prod\limits_{\substack{t_p+1\leqslant u\leqslant t_{p+1}\\ 1\leqslant v\leqslant M}} |h_{uv}^J|}\sigma$$

$$+\frac{1}{t_p}\int\limits_{S(r)} \log|W^\alpha|\sigma - \frac{\varepsilon}{2}T_f(r) - o(T_f(r))\,. \tag{4.19}$$

For each $J \subset \{1,\dots,q\}$, $\#J = n$, take β_J a meromorphic function on $\mathbb{C}^m$ such that

$$\big(\dots;\beta_J b_\ell \widetilde{\psi}_j^J(f):\dots:\beta_J h_{uv}^J:\dots\big) \tag{4.20}$$

is a reduced representation of the meromorphic map

$$g_J := \Big(\dots b_\ell \widetilde{\psi}_j^J(f):\dots:h_{uv}^J\dots\Big)_{\substack{1\leqslant j\leqslant M\\ 1\leqslant \ell\leqslant t_p}}^{\substack{t_p+1\leqslant u\leqslant t_{p+1}\\ 1\leqslant v\leqslant M}} \;:\; \mathbb{C}^m \to \mathbb{C}P^{Mt_{p+1}-1}.$$

By Lemma 2.6 and Lemma 2.7 we have

$$\Big\|\int\limits_{S(r)} \log^+ \frac{|W_J^\alpha|}{\prod\limits_{\substack{1\leqslant j\leqslant M\\ 1\leqslant \ell\leqslant t_p}} |b_\ell \widetilde{\psi}_j^J(f)| \cdot \prod\limits_{\substack{t_p+1\leqslant u\leqslant t_{p+1}\\ 1\leqslant v\leqslant M}} |h_{uv}^J|}\sigma$$

$$= \int\limits_{S(r)} \log^+ \frac{\big|W^\alpha(\dots,\beta_J b_\ell \widetilde{\psi}_j^J(f),\dots,\beta_J h_{uv}^J,\dots)\big|}{\prod\limits_{\substack{1\leqslant j\leqslant M\\ 1\leqslant \ell\leqslant t_p}} |\beta_J b_\ell \widetilde{\psi}_j^J(f)| \cdot \prod\limits_{\substack{t_p+1\leqslant u\leqslant t_{p+1}\\ 1\leqslant v\leqslant M}} |\beta_J h_{uv}^J|}\sigma$$

$$= o(T_{g_J}(r))\,.$$

On the other hand, by Corollary 5.7 in [11] or by Theorem 5.2.29 of [16], we have

$$T_{g_J}(r) \leqslant \sum_{\substack{1\leqslant \ell\leqslant t_p\\ 1\leqslant j\leqslant M}} T_{\frac{b_\ell \widetilde{\psi}_j^J(f)}{b_1 \widetilde{\psi}_1^J(f)}}(r) + \sum_{\substack{t_p+1\leqslant u\leqslant t_{p+1}\\ 1\leqslant v\leqslant M}} T_{\frac{h_{uv}^J}{b_1 \widetilde{\psi}_1^J(f)}}(r) + O(1)$$

$$\leqslant O\Big(\sum_{0\leqslant i\leqslant n} T_{\frac{f_i}{f_0}}(r)\Big) + o(T_f(r)) = O(T_f(r))\,.$$

Hence, for any $J \subset \{1, \dots, q\}$, $\#J = n$, we have

$$\Big\| \int\limits_{S(r)} \log^+ \frac{|W_J^\alpha|}{\prod\limits_{\substack{1 \leqslant j \leqslant M \\ 1 \leqslant \ell \leqslant t_p}} |b_\ell \widetilde{\psi}_j^J(f)| \cdot \prod\limits_{\substack{t_p+1 \leqslant u \leqslant t_{p+1} \\ 1 \leqslant v \leqslant M}} |h_{uv}^J|} \sigma = o(T_f(r))\,.$$

This implies that

$$\Big\| \int\limits_{S(r)} \max_J \log^+ \frac{|W_j^\alpha|}{\prod\limits_{\substack{1 \leqslant j \leqslant M \\ 1 \leqslant \ell \leqslant t_p}} |b_\ell \widetilde{\psi}_j^J(f)| \cdot \prod\limits_{\substack{t_p+1 \leqslant u \leqslant t_{p+1} \\ 1 \leqslant v \leqslant M}} |h_{uv}^J|} \sigma = o(T_f(r))\,. \qquad (4.21)$$

By (4.19) and (4.21) we get

$$\Big\| \int\limits_{S(r)} \min_J \log \prod_{j=1}^M |\psi_j^J(f)| \sigma \geq \frac{1}{t_p} \int\limits_{S(r)} \log |W^\alpha| \sigma - \frac{\varepsilon}{2} T_f(r) - o(T_f(r)).$$

Therefore, by (4.14), we obtain that

$$\Big\| \frac{1}{A \cdot t_p} \int\limits_{S(r)} \log \frac{(\prod_{j=1}^q |Q_j(f)|)^{A \cdot t_p}}{|W^\alpha|} \sigma \geq (q-n-1)d \cdot T_f(r)$$
$$-(\frac{\epsilon}{2} + \frac{\epsilon}{2A}) \cdot T_f(r) - o(T_f(r)). \qquad (4.22)$$

We recall that

$$Q_j = \sum_{I \in \mathcal{T}_d} a_{jI} x^I, \quad a_{jI} \in \mathcal{K}_f \quad (j = 0, \dots, q).$$

Let again $T = (\dots, t_{kI}, \dots)$ $(k \in \{0, \dots, q\}, I \in \mathcal{T}_d)$ be a family of variables and

$$\widetilde{Q}_j = \sum_{I \in \mathcal{T}_d} t_{jI} x^I \in \mathbb{Z}[T, x], \quad j = 0, \dots, q.$$

For each $H \subset \{1, ..., q\}, \#H = (n+1)$, let $\widetilde{R}_H \in \mathbb{Z}[T]$ be the resultant of $\{\widetilde{Q}_j\}_{j \in H}$. By Proposition 2.1 there exists a positive integer s (without loss of generality) common for all H and polynomials $\{\widetilde{b}_{ij}^H\}(0 \leqslant i \leqslant n; j \in H)$ in $\mathbb{Z}[T, x]$, which are zero or homogeneous in x of degree $(s-d)$, such that

$$x_i^s \cdot \widetilde{R}_H = \sum_{j \in H} \widetilde{b}_{ij}^H \cdot \widetilde{Q}_j \quad \text{for all} \quad i \in \{0, \dots, n\},$$

and $R_H = \widetilde{R}_H(\dots, a_{kI}, \dots) \not\equiv 0$. We note again that $R_H \in \mathcal{K}_f$.
Set

$$b_{ij}^H := \widetilde{b}_{ij}^H\big((\dots, a_{jI}, \dots), (x_0, \dots, x_n)\big).$$

Then we have

$$f_i^s \cdot R_H = \sum_{j \in H} b_{ij}^H(f) \cdot Q_j(f) \quad \text{for all } i \in \{0, \dots, n\}.$$

This implies, since $(f_0 : \dots : f_n)$ is a reduced representation, i.e. $\{f_0 = \dots = f_n = 0\}$ of codimension ≥ 2, that

$$\nu_{R_H} \geq \min_{j \in H} \nu_{Q_j(f)} - \sum_{\substack{0 \leqslant i \leqslant n \\ j \in H}} \nu_{\frac{1}{b_{ij}^H(f)}} \tag{4.23}$$

outside an analytic subset of codimension ≥ 2. Then, we have

$$\begin{aligned}
\nu_{\frac{(\prod_{j=1}^q Q_j(f))^{A \cdot t_p}}{W^\alpha}} \leqslant & \max_{J \subset \{1, \dots, q\}, \# J = n} \nu_{\frac{(\prod_{j \in J} Q_j(f))^{A \cdot t_p}}{W^\alpha}} \\
& + \min_{E \subset \{1, \dots, q\}, \# E = q-n} \nu_{(\prod_{j \in E} Q_j(f))^{A \cdot t_p}} \\
\leqslant & \max_{J \subset \{1, \dots, q\}, \# J = n} \nu_{\frac{(\prod_{j \in J} Q_j(f))^{A \cdot t_p}}{W_J^\alpha}} \\
& + At_p \min_{E \subset \{1, \dots, q\}, \# E = q-n} \nu_{\prod_{j \in E} Q_j(f)} \\
\leqslant & \max_{J \subset \{1, \dots, q\}, \# J = n} \nu_{\frac{(\prod_{j \in J} Q_j(f))^{A \cdot t_p}}{W_J^\alpha}} \\
& + (q-n) At_p \sum_{\substack{H \in \{1, \dots, q\} \\ \# H = n+1}} \Big(\nu_{R_H} + \sum_{\substack{0 \leqslant i \leqslant n \\ j \in H}} \nu_{\frac{1}{b_{ij}^H(f)}} \Big)
\end{aligned} \tag{4.24}$$

outside an analytic subset of codimension ≥ 2 (note that

$$W^\alpha_J = W^\alpha(b_\ell\widetilde{\psi}^J_j(f), 1 \leqslant \ell \leqslant t_p, 1 \leqslant j \leqslant M, h^J_{uv}, t_p+1 \leqslant u \leqslant t_{p+1}, 1 \leqslant v \leqslant M)$$
$$= W^\alpha(b_\ell(c^J_{jI^J_j})^{-1}\psi^J_j(f), 1 \leqslant \ell \leqslant t_p, 1 \leqslant j \leqslant M, h^J_{uv}, t_p+1 \leqslant u \leqslant t_{p+1}, 1 \leqslant v \leqslant M)$$
$$= \det C_J \cdot W^\alpha, \quad C_J \in GL(t_{p+1}M, \mathbb{C})).$$

Take $L = Mt_{p+1} - 1 \leqslant \binom{n+N}{n} t_{p_0+1} - 1$ (note that $p \leqslant p_0$). We recall that

$$N = d \cdot [2(n+1)(2^n-1)(nd+1)\epsilon^{-1} + n + 1],$$
$$p_0 = \Big[\frac{\big(\binom{n+N}{n}^2 . \binom{q}{n} - 1\big). \log\big(\binom{n+N}{n}^2 . \binom{q}{n}\big)}{\log(1 + \frac{\epsilon}{2\binom{n+N}{n}N})} + 1\Big]^2, \text{ and}$$
$$t_{p_0+1} \overset{(4.15)}{<} \left(\binom{n+N}{n}^2 . \binom{q}{n} + p_0\right)^{\binom{n+N}{n}^2 . \binom{q}{n} - 1}.$$

Furthermore, in the case of fixed hypersurfaces ($Q_j \in \mathbb{C}[x_0, \dots, x_n], j = 1, \dots, q$), that is in the case of ($Q_j \in \mathbb{C}[x_0, \dots, x_n], j = 1, \dots, q$) with constant coefficients $a_{jI} \in \mathbb{C}$, by the definition of $\mathcal{L}(p)$ we have $t_p = 1$ for all positive integer p, and then $L = M - 1$.

For each $J \subset \{1, \dots, q\}$, $\#J = n$, we use again the reduced representation (see (4.20))

$$\big(\dots; \beta_J b_\ell \widetilde{\psi}^J_j(f) : \dots : \beta_J h^J_{uv} : \dots\big)$$

of the meromorphic map

$$g_J := \Big(\dots b_\ell\widetilde{\psi}^J_j(f) : \dots : h^J_{uv} \dots\Big)^{\substack{t_p+1 \leqslant u \leqslant t_{p+1} \\ 1 \leqslant v \leqslant M}}_{\substack{1 \leqslant j \leqslant M \\ 1 \leqslant \ell \leqslant t_p}} \; : \; \mathbb{C}^m \to \mathbb{C}P^{Mt_{p+1}-1}.$$

For any $J = (j_1, \dots, j_n) \subset \{1, \dots, q\}$, set

$$\nu_J := \nu_{\beta_J} + \nu_{\frac{1}{\beta_J}} + \sum_{\substack{1 \leqslant j \leqslant M \\ 1 \leqslant \ell \leqslant t_p}} \nu_{\frac{1}{b_\ell (c^J_{jI^J_j})^{-1}\gamma^J_j(f)}} + \sum_{\substack{t_p+1 \leqslant u \leqslant t_{p+1} \\ 1 \leqslant v \leqslant M}} \nu_{\frac{1}{h^J_{uv}}}$$

(note that γ^J_j is defined by (4.5)). It is easy to see that $N_{\nu_J}(r) = o(T_f(r))$.

By Proposition 2.8 and Lemmas 2.5-2.6, we have (outside an analytic subset of codimension ≥ 2)

$$\nu_{\frac{\left(\prod_{j\in J} Q_j(f)\right)^{A\cdot t_p}}{W_J^{\alpha}}} + \sum_{\substack{1\leqslant j\leqslant M\\ 1\leqslant \ell\leqslant t_p}} \nu_{b_\ell (c^J_{jI^J_j})^{-1}\gamma^J_j(f)} +$$

$$+ \sum_{\substack{t_p+1\leqslant u\leqslant t_{p+1}\\ 1\leqslant v\leqslant M}} \nu_{h^J_{uv}} - \nu_{\left(\frac{(\prod_{j\in J} Q_j(f))^{A\cdot t_p}}{W_J^{\alpha}}\right)^{-1}} - \nu_J$$

$$\leqslant \nu_{\frac{\left(\prod_{\substack{1\leqslant j\leqslant M\\ 1\leqslant \ell\leqslant t_p}} b_\ell (c^J_{jI^J_j})^{-1}\right)\cdot\left(\prod_{j\in J} Q_j(f)\right)^{A\cdot t_p}\cdot\prod_{\ell=1}^{M}(\gamma^J_\ell(f))^{t_p}\cdot\prod_{\substack{t_p+1\leqslant u\leqslant t_{p+1}\\ 1\leqslant v\leqslant M}} h^J_{uv}}{W_J^{\alpha}}}$$

$$\overset{(4.11)}{=} \nu_{\frac{\prod_{\substack{1\leqslant j\leqslant M\\ 1\leqslant \ell\leqslant t_p}}\left(b_\ell (c^J_{jI^J_j})^{-1}\psi^J_j(f)\right)\prod_{\substack{t_p+1\leqslant u\leqslant t_{p+1}\\ 1\leqslant v\leqslant M}} h^J_{uv}}{W_J^{\alpha}}}$$

$$= \nu_{\frac{\prod_{\substack{1\leqslant j\leqslant M\\ 1\leqslant \ell\leqslant t_p}}\left(\beta_J b_\ell (c^J_{jI^J_j})^{-1}\psi^J_j(f)\right)\prod_{\substack{t_p+1\leqslant u\leqslant t_{p+1}\\ 1\leqslant v\leqslant M}} \beta_J h^J_{uv}}{W^{\alpha}(\ldots,\beta_J b_\ell (c^J_{jI^J_j})^{-1}\psi^J_j(f),\ldots,\beta_J h^J_{uv},\ldots)}}$$

$$\leqslant \sum_{\substack{1\leqslant j\leqslant M\\ 1\leqslant \ell\leqslant t_p}} \min\{\nu_{\beta_J b_\ell (c^J_{jI^J_j})^{-1}\psi^J_j(f)}, L\} + \sum_{\substack{t_p+1\leqslant u\leqslant t_{p+1}\\ 1\leqslant v\leqslant M}} \min\{\nu_{\beta_J h^J_{uv}}, L\}$$

$$\overset{(4.5)}{\leqslant} At_p\sum_{i=1}^{n}\min\{\nu_{Q_{j_i}(f)}, L\} + \sum_{\substack{1\leqslant j\leqslant M\\ 1\leqslant \ell\leqslant t_p}} \nu_{\beta_J b_\ell (c^J_{jI^J_j})^{-1}\gamma^J_j(f)} + \sum_{\substack{t_p+1\leqslant u\leqslant t_{p+1}\\ 1\leqslant v\leqslant M}} \nu_{\beta_J h^J_{uv}}$$

$$\leqslant At_p\sum_{j=1}^{q}\min\{\nu_{Q_j(f)}, L\} + \sum_{\substack{1\leqslant j\leqslant M\\ 1\leqslant \ell\leqslant t_p}} \nu_{b_\ell (c^J_{jI^J_j})^{-1}\gamma^J_j(f)} + \sum_{\substack{t_p+1\leqslant u\leqslant t_{p+1}\\ 1\leqslant v\leqslant M}} \nu_{h^J_{uv}} + O(\nu_{\beta_J}).$$

This implies that for any J (outside an analytic subset of codimension ≥ 2)

$$\nu_{\frac{(\prod_{j\in J} Q_j(f))^{A\cdot t_p}}{W_J^\alpha}} \leqslant At_p\sum_{j=1}^{q}\min\{\nu_{Q_j(f)}, L\} + \nu_{\left(\frac{(\prod_{j\in J} Q_j(f))^{A\cdot t_p}}{W_J^\alpha}\right)^{-1}} + \nu\,, \qquad (4.25)$$

where ν is a divisor on $\mathbb{C}^m$ such that $N_\nu(r) = o(T_f(r))$.

Since the set $\overline{\{\nu_{\left(\frac{(\prod_{j\in J} Q_j(f))^{A\cdot t_p}}{W_J^\alpha}\right)} > 0\}} \cap \overline{\{\nu_{\left(\frac{(\prod_{j\in J} Q_j(f))^{A\cdot t_p}}{W_J^\alpha}\right)^{-1}} > 0\}}$ is an analytic set of codimension ≥ 2, and the terms on the right hand side of (4.25) are all nonnegative, this implies that for all J (outside an analytic subset of codimension ≥ 2)

$$\nu_{\frac{(\prod_{j\in J} Q_j(f))^{A\cdot t_p}}{W_J^\alpha}} \leqslant At_p\sum_{j=1}^{q}\min\{\nu_{Q_j(f)}, L\} + \nu\,.$$

Then, by (4.24) we have

$$\begin{aligned}\nu_{\frac{(\prod_{j=1}^{q} Q_j(f))^{A\cdot t_p}}{W^\alpha}} \leqslant At_p\sum_{j=1}^{q}\min\{\nu_{Q_j(f)}, L\} + \nu \\ + (q-n)At_p\sum_{\substack{H\subset\{1,\dots,q\}\\ \#H=n+1}}\Big(\nu_{R_H} + \sum_{\substack{0\leqslant i\leqslant n\\ j\in H}}\nu_{\frac{1}{b_{ij}^H(f)}}\Big) \qquad (4.26)\end{aligned}$$

outside an analytic subset of codimension ≥ 2.

By Jensen's formula and by (4.26), we get

$$\begin{aligned}
&\int\limits_{S(r)} \log \frac{(\prod_{j=1}^{q} |Q_j(f)|)^{A\cdot t_p}}{|W^\alpha|}\sigma \\
&= N_{\frac{(\prod_{j=1}^{q} Q_j(f))^{A\cdot t_p}}{W^\alpha}}(r) - N_{\left(\frac{(\prod_{j=1}^{q} Q_j(f))^{A\cdot t_p}}{W^\alpha}\right)^{-1}}(r) + O(1) \\
&\leqslant At_p \sum_{j=1}^{q} N_f^{(L)}(r, Q_j) + N_\nu(r) + O(1) \\
&\qquad\qquad + (q-n)At_p \sum_{\substack{H\subset\{1,\dots,q\}\\ \#H=n+1}} \Big(N_{R_H}(r) + \sum_{\substack{0\leqslant i\leqslant n\\ j\in H}} N_{\frac{1}{b_{ij}^H(f)}}(r)\Big) \\
&\leqslant At_p \sum_{j=1}^{q} N_f^{(L)}(r, Q_j) + o(T_f(r))
\end{aligned}$$

(note that $R_H \in \mathcal{K}_f,\ b_{ij}^H \in \mathcal{K}_f[x_0, \dots, x_n]$).
Combining with (4.22), we have

$$\|(q-n-1-\varepsilon)T_f(r) \leqslant \sum_{j=1}^{q} \frac{1}{d} N_f^{(L)}(r, Q_j), \tag{4.27}$$

(note that $A > 1$).

We now prove the theorem for the general case: $\deg Q_j = d_j$. Denote by d the least common multiple of $d_1, \dots, d_q$ and put $d_j^* := \frac{d}{d_j}$. By (4.27) with the moving hypersurfaces $Q_j^{d_j^*}$ $(j \in \{1\dots, q\})$ of common degree d, we have

$$\begin{aligned}
\|(q-n-1-\varepsilon)T_f(r) \leqslant \sum_{j=1}^{q} \frac{1}{d} N_f^{(L)}(r, Q_j^{d_j^*}) &\leqslant \sum_{j=1}^{q} \frac{d_j^*}{d} N_f^{([\frac{L}{d_j^*}+1])}(r, Q_j) \\
&\leqslant \sum_{j=1}^{q} \frac{1}{d_j} N_f^{(L_j)}(r, Q_j),
\end{aligned}$$

where $L_j := [\frac{d_j L}{d} + 1]$. This completes the proof of the Main Theorem. □

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