AN EXTENSION OF THE CARTAN-NOCHKA SECOND MAIN THEOREM FOR HYPERSURFACES

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Abstract

In 1983, Nochka proved a conjecture of Cartan on defects of holomorphic curves in $\mathbb{C}P^n$ relative to a possibly degenerate set of hyperplanes. In this paper, we generalize the Nochka's theorem to the case of curves in a complex projective variety intersecting hypersurfaces in subgeneral position.

1 Introduction and statements

Let f be a holomorphic mapping of $\mathbb{C}$ into $\mathbb{C}P^M$, with a reduced representation $f = (f_0 : \cdots : f_M)$. The characteristic function $T_f(r)$ of f is defined by

$$T_f(r) := \frac{1}{2\pi}\int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta,$$

where $\|f\| := \max\{|f_0|, \ldots, |f_M|\}$.

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Let L be a positive integer or $+\infty$, and let ν be a divisor on $\mathbb{C}$. Set $\nu^{[L]}(z) := \min\{\nu(z), L\}$. The truncated counting function to level L of ν is defined by

$$N_\nu^{[L]}(r) := \int_1^r \frac{\sum_{|z|<t} \nu^{[L]}(z)}{t} dt \quad (1 < r < +\infty).$$

Let φ be a nonzero meromorphic function on $\mathbb{C}$. Denote by ν_φ be the zero divisor of φ. Set $N_\varphi^{[L]}(r) := N_{\nu_\varphi}^{[L]}(r)$.

Let D be a hypersurface in $\mathbb{C}P^M$ of degree $d \geq 1$. Let $Q \in \mathbb{C}[x_0, \ldots, x_M]$ be a homogeneous polynomial of degree d defining D. Set $\nu_D^{[L]}(f) := \nu_{Q(f)}^{[L]}$, and $N_f^{[L]}(r, D) := N_{Q(f)}^{[L]}(r)$. For brevity we will omit the character $^{[L]}$ in the counting function and in the divisor if $L = +\infty$.

For the holomorphic function φ, we have the following Jensen's formula

$$N_\varphi(r) = \int_0^{2\pi} \log |\varphi(re^{i\theta})| \frac{d\theta}{2\pi} + O(1).$$

Let $V \subset \mathbb{C}P^M$ be a smooth complex projective variety of dimension $n \geq 1$. Let $D_1, \ldots, D_k$ $(k \geq 1)$ be hypersurfaces in $\mathbb{C}P^M$ of degree d_j. The hypersurfaces $D_1, \ldots, D_k$ are said to be in *general position in V* if for any distinct indices $1 \leqslant i_1 < \cdots < i_s \leqslant k$, $(1 \leqslant s \leqslant n+1)$, there exist hypersurfaces $D'_1, \ldots, D'_{n+1-s}$ in $\mathbb{C}P^M$ such that

$$V \cap D_{i_1} \cap \cdots \cap D_{i_s} \cap D'_1 \cap \cdots \cap D'_{n+1-s} = \varnothing$$

(see Noguchi-Winkelmann [14] and Ru [17] for similar definitions). In particular for hypersurfaces $D_1, \ldots, D_k$ in general position in V, we have $V \not\subseteq D_j$ for all $j = 1, ..., k$. By definition, we also call an empty set of hypersurfaces in $\mathbb{C}P^M$ to be in general position in V.

Definition 1.1. *Let $N \geq n$ and $q \geq N+1$. Hypersurfaces $D_1, \ldots, D_q$ in $\mathbb{C}P^M$ with $V \not\subseteq D_j$ for all $j = 1, ..., q$ are said to be in N-subgeneral position in V if the two following conditions are satisfied:*

(i) For every $1 \leqslant j_0 < \cdots < j_N \leqslant q$, $V \cap D_{j_0} \cap \cdots \cap D_{j_N} = \varnothing$.

(ii) For any subset $J \subset \{1, \ldots, q\}$ such that $0 < \#J \leqslant n+1$ and $\{D_j,\ j \in J\}$ are in general position in V and $V \cap (\cap_{j\in J} D_j) \neq \emptyset$, there exists an irreducible component σ_J of $V \cap (\cap_{j\in J} D_j)$ with $\dim \sigma_J = \dim \big(V \cap (\cap_{j\in J} D_j)\big)$ such that for any $i \in \{1, \ldots, q\} \setminus J$, if $\dim \big(V \cap (\cap_{j\in J} D_j)\big) = \dim \big(V \cap D_i \cap (\cap_{j\in J} D_j)\big)$, then D_i contains σ_J.

We first remark that if $V = \mathbb{C}P^M$ is a complex projective space and $\{D_j\}_{j=1}^q$ are hyperplanes, then the condition (ii) in the above definition is automatically satisfied. We also note that in the case where $N = n$, the condition (i) implies the condition (ii). Therefore, in this case the above definition coincides with the concept of general position.

We finally construct an example of hypersurfaces in $(n+1)$-subgeneral position in V, which are, however, not in general position in V: Let $D_1, \ldots, D_q$ $(q \geq n+1)$ be hypersurfaces in $\mathbb{C}P^M$ in general position in V. Let $\{J_1, \ldots, J_K\}$ $(K = \binom{q}{n})$ be the set of all subsets J of $\{1, \ldots, q\}$ such that $\#J = n$. It is clear that $0 < \#\big(V \cap (\cap_{j\in J_i} D_j)\big) < \infty$ for all $1 \leqslant i \leqslant K$. We define hypersurfaces $D_{t_1}, \ldots, D_{t_K}$ in $\mathbb{C}P^m$ by induction as follows: Take a hypersurface D_{t_1} passing through a point $A_1 \in V \cap (\cap_{j\in J_1} D_j)$, but not containing any irreducible component σ of $V \cap (\cap_{j\in J} D_j)$ with $\dim \sigma = \dim \big(V \cap (\cap_{j\in J} D_j)\big)$ for all $J_1 \neq J \subset \{1, \ldots, q\}$ with $0 < \#J \leqslant n$ (note that the number of these irreducible components σ is finite, and $\{A_1\} \neq \sigma$, since $D_1, \ldots, D_q$ are in general position in V). Then, for any $\varnothing \neq J \subset \{1, \ldots, q, t_1\}$, $\#J \leqslant n+1$, $J \neq J_1 \cup \{t_1\}$, the hypersurfaces $\{D_j,\ j \in J\}$ are in general position in V. Assume that hypersurfaces $D_{t_1}, \ldots, D_{t_{i-1}}$ $(2 \leqslant i \leqslant K)$ in $\mathbb{C}P^M$ are chosen, we next choose a hypersurface D_{t_i} in $\mathbb{C}P^M$ passing through a point $A_i \in V \cap (\cap_{j\in J_i} D_j)$, but not containing any irreducible component σ of $V \cap (\cap_{j\in J} D_j)$ with $\dim \sigma = \dim \big(V \cap (\cap_{j\in J} D_j)\big)$ for any $J_i \neq J \subset \{1, \ldots, q, t_1, \ldots, t_{i-1}\}$ with $0 < \#J \leqslant n$ (note that $\{A_i\} \neq \sigma$ since $\{D_j,\ j \in J'\}$ are in general position in V for all $J' \subset \{1, \ldots, q, t_1, \ldots, t_{i-1}\}$, $0 < \#J' \leqslant n+1$, $J' \neq J_s \cup \{t_s\}$ $(s = 1, \ldots, i-1)$). By our choices of the D_{t_i}'s, for any $J \subset \{1, \ldots, q, t_1, \ldots, t_K\}, \#J \leqslant n+1$, the hypersurfaces $\{D_j,\ j \in J\}$ are in general position in V except in the cases $J = J_i \cup \{t_i\}$ $(i = 1, \ldots, K)$. Therefore for any $\varnothing \neq J \subset \{1, \ldots, q, t_1, \ldots, t_K\}$, $\#J \leqslant n+1$ such that $\{D_j,\ j \in J\}$ are in general position in V and for any $i \in \{1, \ldots, q, t_1, \ldots, t_K\} \setminus J$, we have that $\dim \big(V \cap D_i \cap (\cap_{j\in J} D_j)\big) = \dim \big(V \cap (\cap_{j\in J} D_j)\big)$ if and only if either $\#J = n+1$ and then $V \cap (\cap_{j\in J} D_j) = \varnothing$ or there exists $k \in \{1, \ldots, K\}$ such that $\{i\} \cup J = \{t_k\} \cup J_k$. This implies that for $N = n+1$, the hypersurfaces $D_1, \ldots, D_q, D_{t_1}, \ldots, D_{t_K}$ satisfy the conditions (i) and (ii) of Definition 1.1, and, hence, they are in $(n+1)$-subgeneral position. But, they are not in general position.

In 1933, Cartan [2] proved the Second Main Theorem for linearly nondegenerate holomorphic mappings of $\mathbb{C}$ into $\mathbb{C}P^n$ intersecting hyperplanes

in general position. He also proposed a conjecture for the case where the hyperplanes are only in subgeneral position. This conjecture was solved by Nochka [13].
As usual, by the notation "$\|P$" we mean the assertion P holds for all $r \in [1, +\infty)$ excluding a Borel subset E of $(1, +\infty)$ with $\int_E dr < +\infty$.

Theorem 1.2 (Nochka). *Let f be a linearly nondegenerate holomorphic mapping of $\mathbb{C}$ into $\mathbb{C}P^n$ and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{C}P^n$ in N-subgeneral position, where $N \geq n$ and $q \geq 2N - n + 1$. Then, for every $\epsilon > 0$,*

$$\Big\|(q - 2N + n - 1 - \epsilon)T_f(r) \leqslant \sum_{j=1}^{q} N_f^{[n]}(r, H_j).$$

Recently, the Second Main Theorem for the case of hypersurfaces in general position was established by Ru ([16], [17]), see also Dethloff and Tan [5]. For the case where hypersurfaces are not in general position, in [21] Thai and Thu obtained a Second Main Theorem for algebraically non-degenerate holomorphic maps $f : \mathbb{C} \to \mathbb{C}P^k \subset \mathbb{C}P^n$, without truncated multiplicities, and for a special class of hypersurfaces in $\mathbb{C}P^n$.

In 2009, Ru [17] proved that

Theorem 1.3. *Let $V \subset \mathbb{C}P^M$ be a smooth complex projective variety of dimension $n \geq 1$. Let f be an algebraically nondegenerate holomorphic mapping of $\mathbb{C}$ into V. Let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{C}P^M$ of degree d_j, in general position in V. Then for every $\epsilon > 0$,*

$$\Big\|(q - n - 1 - \epsilon)T_f(r) \leqslant \sum_{j=1}^{q} \frac{1}{d_j} N(r, D_j).$$

Motivated by the case of hyperplanes, in this paper we prove the following Second Main Theorem for hypersurfaces being in N-subgeneral position.

Theorem 1.4. *Let $V \subset \mathbb{C}P^M$ be a smooth complex projective variety of dimension $n \geq 1$. Let f be an algebraically nondegenerate holomorphic mapping of $\mathbb{C}$ into V. Let $D_1, \ldots, D_q$ ($V \not\subseteq D_j$) be hypersurfaces in $\mathbb{C}P^M$ of degree d_j, in N-subgeneral position in V, where $N \geq n$ and $q \geq 2N - n + 1$. Then,* *for every* $\epsilon > 0$, *there exist positive integers* L_j $(j = 1, ..., q)$ *depending on* $n, \deg V, N, d_j$ $(j = 1, ..., q), q, \epsilon$ *in an explicit way such that*

$$\Big\|(q - 2N + n - 1 - \epsilon)T_f(r) \leqslant \sum_{j=1}^{q} \frac{1}{d_j} N_f^{[L_j]}(r, D_j). \tag{1.1}$$

The explicit bounds which we will get with the proof of Theorem 1.4 are as follows:

Proposition 1.5. *Assume without loss of generality that* $\epsilon \leqslant 1$. *Let* d *be the least common multiple of the* d_j*'s. Put*

$$m_0 = m_0(n, \deg V, N, d, q, \epsilon) := [4d^{n+1}q(2n+1)(2N-n+1)\deg V \cdot \frac{1}{\epsilon}] + 1\,,$$

then

$$L_j \leqslant \Big[\frac{d_j\big(\binom{q+m_0-1}{m_0} - 1\big)}{d} + 1\Big]\,, \tag{1.2}$$

where we denote $[x] := \max\{k \in \mathbb{Z} : k \leqslant x\}$ *for a real number* x.

The proof of Theorem 1.4 consists of three parts: In the first part (chapter 2), we extend the Nochka weights from the case of hyperplanes to the case of hypersurfaces. In the second part (chapter 4 until (4.18)) we reduce the case of hypersurfaces to the case of hyperplanes. The method in this part is based on the work of Evertse - Ferretti [8], Nochka [13], and Ru [17]. In the last part, we obtain an effective truncation for the counting functions. For this we devellop a new method using Hilbert weights, which is, in particular, different from the method which is used in the case of nondegenerate holomorphic curves in a complex projective space (see Dethloff-Tan [5]).

We also note that the proof of our Second Main Theorem remains valid if more generally the hypersurfaces have Nochka weights.

Let us finally give an example for the special case $V = \mathbb{C}P^2$. We consider three quadrics $\Gamma_1, \Gamma_2, \Gamma_3$ in $\mathbb{C}P^2$ such that they have one common point A_1. Let A_2, A_3 be distinct points in $\mathbb{C}P^2 \setminus \cup_{i=1}^{3}\Gamma_i$. Let $B_i \in \Gamma_i \setminus (\Gamma_u \cup \Gamma_v)$ $(\{i, u, v\} = \{1, 2, 3\})$ such that the lines B_iA_2, B_iA_3 are distinct and do not pass through any intersection point of any pair of curves in $\cup_{1\leqslant s\leqslant i-1}\{B_sA_2, B_sA_3\}\cup\{\Gamma_1,\Gamma_2,\Gamma_3\}$. Take three distinct lines L_1, L_2, L_3 which do not pass through any intersection point of any pair of curves in $\cup_{1\leqslant i\leqslant 3}\{B_iA_2, B_iA_3\}\cup\{\Gamma_1,\Gamma_2,\Gamma_3\}$ and L_1, L_2, L_3 have the common point A_4 which does not belong to any Γ_i, B_iA_2, B_iA_3 $(i = 1, 2, 3)$. Set $\mathcal{G}_1 := \{\Gamma_1,\Gamma_2,\Gamma_3\}$, $\mathcal{G}_i := \{A_iB_1, A_iB_2, A_iB_3\}$ $(i = 2, 3)$, and $\mathcal{G}_4 := \{L_1, L_2, L_3\}$. Then the curves in the set $\mathcal{G} := \cup_{i=1}^4\mathcal{G}_i$ are in 3-subgeneral position in $\mathbb{C}P^2$. Hence, by Theorem 1.4, for any algebraically nondegenerate holomorphic curve f in $\mathbb{C}P^2$ and for any $\epsilon > 0$,

$$\Big\|(7-\epsilon)T_f(r)\leqslant\sum_{D\in\mathcal{G}}\frac{1}{\deg D}N_f(r, D).$$

On the other hand, we can not get the above inequality from the Second Main Theorem for hypersurfaces in general position (Theorem 1.3). In fact, for any $\mathcal{G}'\subset\mathcal{G}$ such that the curves in $\mathcal{G}'$ are in general position, it is clear that $\#(\mathcal{G}'\cap\mathcal{G}_i)\leqslant 2$ for all $1\leqslant i\leqslant 4$. So, $\#\mathcal{G}'\leqslant 8$. We write $\mathcal{G} = \cup_{i=1}^s\mathcal{G}_i$, such that $\mathcal{G}_i\cap\mathcal{G}_j = \varnothing$ $(1\leqslant i < j\leqslant s)$ and for any $i\in\{1,\ldots,s\}$ the curves in $\mathcal{G}_i$ are in general position. We have $\#\mathcal{G}_1+\cdots+\#\mathcal{G}_s = 12$ and $\#\mathcal{G}_i\leqslant 8$, $(i = 1,\ldots,s)$. By Theorem 1.3, we get

$$\Big\|(\#\mathcal{G}_i-3-\epsilon)T_f(r)\leqslant\sum_{D\in\mathcal{G}_i}\frac{1}{\deg D}N_f(r, D), (i = 1,\ldots s).$$

So by summing up over any partition of $\mathcal{G} = \cup_{i=1}^s\mathcal{G}_i$, since such a partition must have at least two elements, we get at most a term $\Big\|(6-\epsilon)T_f(r)$ on the left hand side.

2 Nochka weights for hypersurfaces in subgeneral position

In this section, we shall prove the existence of the Nochka weights for hypersurfaces in subgeneral position which was proved by Nochka for the case of hyperplanes. We mainly follow the ideas of Chen [3], Nochka [13], Ru-Wong [18], and Vojta [24]. However, we have to pass some difficulties due to the fact that their methods are based on properties of linear algebra. We finally would like to remark that the existence of Nochka weights for the case of infinitely many hyperplanes has been established by N. Toda [22].

Let $V \subset \mathbb{C}P^M$ be a smooth projective variety of dimension n. Throughout of this section, we consider q hypersurfaces $D_1, \ldots, D_q \subset \mathbb{C}P^M$ in N-subgeneral position in V, where $N \geq n$ and $q \geq N+1$. Set $Q := \{1, \ldots, q\}$, $\operatorname{codim}\varnothing := n+1$, $c(\varnothing) := 0$, and $c(R) := \operatorname{codim}\big(V \cap (\cap_{j\in R} D_j)\big)$, where the codimension is taken with respect to V and $\varnothing \neq R \subseteq Q$. It is easy to see that

Remark 2.1. (i) *For any $K \subseteq Q$, we have $c(K) \leqslant \#K$, moreover, $c(K) = \#K$ if and only if $\#K \leqslant n+1$ and the hypersurfaces D_j $(j \in K)$ are in general position in V.*

(ii) *For $K \subseteq K' \subseteq Q$, if $c(K') = \#K'$ then $c(K) = \#K$.*

Lemma 2.2. *Let $K \subseteq R \subseteq Q$ such that $\#K = c(K)$. Then there exists a set K' such that $K \subseteq K' \subseteq R$ and $c(K') = \#K' = c(R)$.*

Proof. We have $\#K = c(K) \leqslant c(R)$. If $c(K) = c(R)$, then the lemma is trivial by taking $K' = K$. If $c(K) < c(R)$, by induction, it suffices to show that there exists $i \in R \setminus K$ such that $c(K \cup \{i\}) = \#K + 1$ $(= c(K) + 1)$.

Suppose that $c(K \cup \{i\}) \neq \#K + 1 = c(K) + 1$ for every $i \in R \setminus K$. Then $c(K \cup \{i\}) = c(K)$ for all $i \in R$. If $K = \varnothing$ this is a contradiction, either, in the case $R \neq \varnothing$, to the hypothesis of N-subgeneral position (including $V \not\subset D_i$), or, if $R = \varnothing$, to the hypothesis $c(K) < c(R)$. If $K \neq \varnothing$ this means that $\dim\big(V \cap D_i \cap (\cap_{j\in K} D_j)\big) = \dim\big(V \cap (\cap_{j\in K} D_j)\big)$ for all $i \in R$. Therefore, since $\{D_j,\ j \in Q\}$ are in N-subgeneral position, there exists an irreducible component σ_K of $V \cap (\cap_{j\in K} D_j)$ with $\dim \sigma_K = \dim\big(V \cap (\cap_{j\in K} D_j)\big)$ such that D_i contains σ_K for all $i \in R \setminus K$. Hence, we get $\dim\big(V \cap (\cap_{j\in R} D_j)\big) = \dim\big(V \cap (\cap_{j\in K} D_j)\big)$. This means that $c(R) = c(K)$. This is a contradiction. This completes the proof of Lemma 2.2. □

Lemma 2.3. (i) *For any subsets $R_1, R_2 \subseteq Q$, we have*

$$c(R_1 \cup R_2) + c(R_1 \cap R_2) \leqslant c(R_1) + c(R_2).$$

(ii) *For any $S_1 \subseteq S_2 \subseteq Q$, we have $\#S_1 - c(S_1) \leqslant \#S_2 - c(S_2)$. Furthermore, if $\#S_2 \leqslant N+1$ then $\#S_2 - c(S_2) \leqslant N - n$.*

Proof. Proof of (i): By Lemma 2.2, there exist subsets K, K_1, K_3 with $K \subseteq R_1 \cap R_2$, $K \subseteq K_1 \subseteq R_1$, $K_1 \subseteq K_3 \subseteq R_1 \cup R_2$, such that

$$\#K = c(K) = c(R_1 \cap R_2),\ \#K_1 = c(K_1) = c(R_1),\ \text{and}$$
$$\#K_3 = c(K_3) = c(R_1 \cup R_2).$$

Set $K_2 := K_3 \setminus K_1$. Then $K_2 \subseteq R_2$. Indeed, otherwise there exists $i \in K_2 \setminus R_2$. Then $i \in R_1 \setminus K_1$. Therefore, $K_3 \supseteq K_1 \cup \{i\} \subseteq R_1$. This implies that $c(R_1) \geq c(K_1 \cup \{i\}) = \#(K_1 \cup \{i\}) = c(K_1) + 1 = c(R_1) + 1$ by Remark 2.1. This is a contradiction. Hence, $K_2 \subseteq R_2$. Therefore, $K_2 \cup K \subseteq R_2$. On the other hand $K_2 \cup K \subseteq K_3$, and $K_2 \cap K \subseteq K_2 \cap K_1 = (K_3 \setminus K_1) \cap K_1 = \varnothing$. From these facts and by Remark 2.1 (ii) we get $c(R_2) \geq c(K_2 \cup K) = \#(K_2 \cup K) = \#K_2 + \#K = (\#K_3 - \#K_1) + \#K = c(R_1 \cup R_2) - c(R_1) + c(R_1 \cap R_2)$. Hence, the assertion (i) holds.

Proof of *(ii)*: By Lemma 2.2, there exist S'_v $(v = 1, 2)$ such that $S'_v \subseteq S_v$, $S'_1 \subseteq S'_2$ and $\#S'_v = c(S'_v) = c(S_v)$. We have $(S'_2 \setminus S'_1) \cap S_1 = \varnothing$. Indeed, otherwise there exists $i \notin S'_1$ such that $S'_2 \supseteq S'_1 \cup \{i\} \subseteq S_1$. Therefore, by Remark 2.1 (ii) we get $c(S_1) + 1 = \#S'_1 + 1 = c(S'_1 \cup \{i\}) \leqslant c(S_1)$. This is a contradiction. Hence, $(S'_2 \setminus S'_1) \cap S_1 = \varnothing$. Thus we have $S'_2 \setminus S'_1 \subseteq S_2 \setminus S_1$. Therefore, $c(S_2) - c(S_1) = \#S'_2 - \#S'_1 = \#(S'_2 \setminus S'_1) \leqslant \#(S_2 \setminus S_1) = \#S_2 - \#S_1$.

If $\#S_2 \leqslant N + 1$, then we choose S_3 such that $S_2 \subseteq S_3 \subseteq Q$ and $\#S_3 = N + 1$. Since D_j $(j \in Q)$ are in N-subgeneral position, we have $c(S_3) = n + 1$. Therefore, $\#S_2 - c(S_2) \leqslant \#S_3 - c(S_3) = N - n$. □

For $R_1 \subsetneq R_2 \subseteq Q$, we set $\rho(R_1, R_2) = \frac{c(R_2) - c(R_1)}{\#R_2 - \#R_1}$. Then by Lemma 2.3, we have $0 \leqslant \rho(R_1, R_2) \leqslant 1$.

Lemma 2.4. *Let $D_1, \ldots, D_q$ be hypersurfaces in N-subgeneral position in V, where $N \geq n$ and $q \geq 2N - n + 1$. Then, there exists a sequence of subsets $\varnothing := R_0 \subsetneq R_1 \subsetneq \cdots \subsetneq R_s \subseteq Q := \{1, \ldots, q\}$ $(s \geq 0)$ satisfying the following conditions:*

(i) $c(R_s) < n + 1$,

(ii) $0 < \rho(R_0, R_1) < \rho(R_1, R_2) < \cdots < \rho(R_{s-1}, R_s) < \frac{n+1-c(R_s)}{2N-n+1-\#R_s}$,

(iii) for any R with $R_{i-1} \subsetneq R \subseteq Q$ $(1 \leqslant i \leqslant s)$, and $c(R_{i-1}) < c(R) < n + 1$, we have that $\rho(R_{i-1}, R_i) \leqslant \rho(R_{i-1}, R)$ and, moreover, if $\rho(R_{i-1}, R_i) = \rho(R_{i-1}, R)$ then $\#R \leqslant \#R_i$.

(iv) for any R with $R_s \subsetneq R \subseteq Q$, if $c(R_s) < c(R) < n+1$, then $\rho(R_s, R) \geq \frac{n+1-c(R_s)}{2N-n+1-\#R_s}$.

Proof. We start the proof by setting $R_0 = \varnothing$. It suffices to show that, under the assumption that there is a sequence $\varnothing := R_0 \subsetneq R_1 \subsetneq \cdots \subsetneq R_s \subseteq Q$ satisfying conditions (i), (ii) and (iii), it satisfies also the condition (iv) or, otherwise, there exists a subset R_{s+1} such that the sequence $\varnothing := R_0 \subsetneq R_1 \subsetneq \cdots \subsetneq R_{s+1} \subseteq Q := \{1, \ldots, q\}$ satisfies conditions (i), (ii) and (iii). In fact, if the latter case occurs, we can reach the desired conclusion after finitely many repetitions of these constructions.

We now consider a sequence $\varnothing := R_0 \subsetneq R_1 \subsetneq \cdots \subsetneq R_s \subseteq Q$ satisfying condition (i), (ii) and (iii). Assume that this sequence does not satisfy the condition (iv). Set $\mathcal{R} := \{R : \ R_s \subsetneq R \subseteq Q, c(R_s) < c(R) < n+1, \ \text{and} \ \rho(R_s, R) < \frac{n+1-c(R_s)}{2N-n+1-\#R_s}\}$. Then, we have $\mathcal{R} \neq \varnothing$. Set $\rho := \min\{\rho(R_s, R) : \ R \in \mathcal{R}\}$. We choose a set R_{s+1} in $\mathcal{R}$ such that $\rho(R_s, R_{s+1}) = \rho$ and $\#R_{s+1}$ is as big as possible.

We now prove that the sequence $R_0 \subsetneq R_1 \subsetneq \cdots \subsetneq R_{s+1}$ satisfies conditions (i), (ii) and (iii).

$*$ It is clear that $c(R_{s+1}) < n+1$, since $R_{s+1} \in \mathcal{R}$.

$*$ If $s \geq 1$, we have $R_{s-1} \subsetneq R_{s+1} \subseteq Q, \ c(R_{s-1}) \leqslant c(R_s) < c(R_{s+1}) < n+1$, and $\#R_{s+1} > \#R_s$. Therefore, since the sequence $R_0 \subsetneq \cdots \subsetneq R_s$ satisfies the condition (iii), we have

$$\rho(R_{s-1}, R_s) < \rho(R_{s-1}, R_{s+1}). \tag{2.1}$$

On the other hand, for any $0 \leqslant a \leqslant c, \ 0 < b < d$ such that $\frac{a}{b} < \frac{c}{d}$, we have

$$\frac{a}{b} < \frac{c-a}{d-b}. \tag{2.2}$$

Therefore, by (2.1) we have $\rho(R_{s-1}, R_s) < \rho(R_s, R_{s+1})$. And if $s = 0$, then we have $\rho(R_0, R_1) = \rho(\varnothing, R_1) = \frac{c(R_1)}{\#R_1} > 0$.
Since $R_{s+1} \in \mathcal{R}$, we get $\rho(R_s, R_{s+1}) = \frac{c(R_{s+1})-c(R_s)}{\#R_{s+1}-\#R_s} < \frac{n+1-c(R_s)}{2N-n+1-\#R_s}$. Hence, in both cases, by using the property (2.2), we get $\rho(R_s, R_{s+1}) < \frac{n+1-c(R_{s+1})}{2N-n+1-\#R_{s+1}}$ (observing that, by the hypothesis of N-subgeneral position in V, we get from $c(R_{s+1}) < n+1$ that $\#R_{s+1} \leqslant N < 2N-n+1$).

$*$ Let R (if there exists any) such that $R_s \subsetneq R \subseteq Q$ and $c(R_s) < c(R) < n+1$. If $\rho(R_s, R) \geq \frac{n+1-c(R_s)}{2N-n+1-\#R_s}$, then $\rho(R_s, R_{s+1}) = \rho < \rho(R_s, R)$. If $\rho(R_s, R) < \frac{n+1-c(R_s)}{2N-n+1-\#R_s}$, then $R \in \mathcal{R}$. Therefore, by our choice of R_{s+1} we have that $\rho(R_s, R_{s+1}) \leqslant \rho(R_s, R)$, furthermore, if $\rho = \rho(R_s, R_{s+1}) = \rho(R_s, R)$ then $\#R \leqslant \#R_{s+1}$.

From theses facts, we get that the sequence $R_0 \subsetneq R_1 \subsetneq \cdots \subsetneq R_{s+1}$ satisfies conditions (i), (ii) and (iii). This completes the proof of Lemma 2.4. □

Proposition 2.5. *Let $D_1, \dots, D_q$ be hypersurfaces in N-subgeneral position in V, where $N \geq n$ and $q \geq 2N - n + 1$. Then, there exist constants $\omega(1), \dots, \omega(q)$ and Θ satisfying the following conditions:*

(i) $0 < \omega(j) \leqslant \Theta \leqslant 1$ $(1 \leqslant j \leqslant q)$,

(ii) $\sum_{j=1}^{q} \omega(j) = \Theta(q - 2N + n - 1) + n + 1$,

(iii) $\frac{n+1}{2N-n+1} \leqslant \Theta \leqslant \frac{n+1}{N+1}$,

(iv) if $R \subseteq Q$ *and* $0 < \#R \leqslant N + 1$, *then* $\sum_{j \in R} \omega(j) \leqslant c(R)$.

Proof. If $N = n$, then $\omega(1) = \cdots = \omega(q) = 1$ and $\Theta = 1$ satisfy the conditions (i) to (iv). Assume that $N > n$. Let $\{R_i\}_{i=0}^{s}$ be a sequence of subsets of $Q := \{1, \dots, q\}$ satisfying the conditions (i) to (iv) of Lemma 2.4. By Lemma 2.4 (i) and by the "N-subgeneral position" condition, we have

$$\#R_s \leqslant N. \tag{2.3}$$

Take a subset R_{s+1} of Q such that $\#R_{s+1} = 2N - n + 1 \geq N + 1$ and, hence, $R_s \subsetneq R_{s+1}$. Then we have $c(R_{s+1}) = n + 1$.

Set

$$\Theta := \rho(R_s, R_{s+1}) = \frac{n + 1 - c(R_s)}{2N - n + 1 - \#R_s}, \text{ and}$$

$$\omega(j) := \begin{cases} \rho(R_i, R_{i+1}) & \text{if } j \in R_{i+1} \setminus R_i \text{ for some } i \text{ with } 0 \leqslant i \leqslant s, \\ \Theta & \text{if } j \notin R_{s+1}. \end{cases}$$

By Lemma 2.4 (ii), $\{\omega(j)\}_{j=1}^{q}$ and Θ satisfy the condition (i) of Proposition 2.5.

We have

$$\begin{aligned} \sum_{j=1}^{q} \omega(j) &= \sum_{j \in Q \setminus R_{s+1}} \omega(j) + \sum_{i=0}^{s} \sum_{j \in R_{i+1} \setminus R_i} \omega(j) \\ &= \Theta(q - 2N + n - 1) + \sum_{i=0}^{s} \big(c(R_{i+1}) - c(R_i)\big) \\ &= \Theta(q - 2N + n - 1) + c(R_{s+1}) \\ &= \Theta(q - 2N + n - 1) + n + 1. \end{aligned}$$

Thus, $\{\omega(j)\}_{j=1}^{q}$ and Θ satisfy the condition (ii) of Proposition 2.5.

We next check the condition (iii). By (i) and (ii), we have

$$n + 1 = \sum_{j=1}^{q} \omega(j) - \Theta(q - 2N + n - 1) \leqslant \Theta(2N - n + 1).$$

By Lemma 2.3 (ii) we have

$$\Theta = \frac{n+1-c(R_s)}{N+1+(N-n-\#R_s)} \leqslant \frac{n+1-c(R_s)}{N+1-c(R_s)} \leqslant \frac{n+1}{N+1}.$$

Finally we check the condition (iv). Take an arbitrary subset R of Q with $0 < \#R \leqslant N+1$.

Case 1: $c(R \cup R_s) \leqslant n$.

Set

$$R'_i := \begin{cases} R \cap R_i & \text{if } 0 \leqslant i \leqslant s, \\ R & \text{if } i = s+1. \end{cases}$$

We now prove that: for any $i \in \{1, \ldots, s+1\}$, if $\#R'_i > \#R'_{i-1}$ then

$$c(R'_i \cup R_{i-1}) > c(R_{i-1}) \tag{2.4}$$

and

$$\rho(R_{i-1}, R_i) \leqslant \rho(R'_{i-1}, R'_i). \tag{2.5}$$

∗ If $i = 1$ then $c(R'_1 \cup R_0) = c(R'_1) > 0 = c(R_0)$ (note that $R'_1 \neq \varnothing$, since $\#R'_1 > \#R'_0 = 0$).

∗ If $i \geq 2$, then since $\#R'_i > \#R'_{i-1}$, we have $\#(R'_i \cup R_{i-1}) > \#R_{i-1}$. On the other hand $c(R_{i-2}) < c(R_{i-1}) \leqslant c(R'_i \cup R_{i-1}) \leqslant c(R \cup R_s) \leqslant n$ (note that $\rho(R_{i-2}, R_{i-1}) > 0$). Therefore, by Lemma 2.4, (iii) we have $\rho(R_{i-2}, R_{i-1}) < \rho(R_{i-2}, R'_i \cup R_{i-1})$. This means that

$$\frac{c(R_{i-1}) - c(R_{i-2})}{\#R_{i-1} - \#R_{i-2}} < \frac{c(R'_i \cup R_{i-1}) - c(R_{i-2})}{\#(R'_i \cup R_{i-1}) - \#R_{i-2}}.$$

Therefore, since $\#R_{i-1} < \#(R'_i \cup R_{i-1})$, we have $c(R_{i-1}) < c(R'_i \cup R_{i-1})$. We get (2.4).

We next prove (2.5). By (2.4), we have $c(R_{i-1}) < c(R'_i \cup R_{i-1}) \leqslant c(R \cup R_s) \leqslant n$. Hence, by Lemma 2.4, (iii) for the case $1 \leqslant i \leqslant s$ and (iv) for the case $i = s+1$, we have

$$\rho(R_{i-1}, R_i) \leqslant \rho(R_{i-1}, R'_i \cup R_{i-1}) \text{ for all } i \in \{1, \ldots, s+1\},$$

(note that $\rho(R_s, R_{s+1}) = \frac{n+1-c(R_s)}{2N-n+1-\#R_s}$).

Therefore, by Lemma 2.3, (i) we have

$$\begin{aligned}\rho(R_{i-1}, R_i) &\leqslant \rho(R_{i-1}, R'_i \cup R_{i-1}) \\ &= \frac{c(R'_i \cup R_{i-1}) - c(R_{i-1})}{\#(R'_i \cup R_{i-1}) - \#R_{i-1}} \leqslant \frac{c(R'_i) - c(R'_i \cap R_{i-1})}{\#(R'_i \cup R_{i-1}) - \#R_{i-1}} \\ &= \frac{c(R'_i) - c(R'_{i-1})}{\#R'_i - \#(R'_i \cap R_{i-1})} = \frac{c(R'_i) - c(R'_{i-1})}{\#R'_i - \#R'_{i-1}} \\ &= \rho(R'_{i-1}, R'_i),\end{aligned}$$

(note that $R'_{i-1} = R'_i \cap R_{i-1}$). We get (2.5).

By (2.5), we get that

$$\omega(j) \leqslant \rho(R'_{i-1}, R'_i) \text{ for all } j \in R'_i \setminus R'_{i-1} \ (1 \leqslant i \leqslant s+1). \tag{2.6}$$

In fact, for $j \in R'_{s+1} \setminus R'_s$ we have $\omega(j) \leqslant \Theta = \rho(R_s, R_{s+1}) \leqslant \rho(R'_s, R'_{s+1})$, and for $j \in R'_i \setminus R'_{i-1} \subseteq R_i \setminus R_{i-1}$ $(1 \leqslant i \leqslant s)$ we have $\omega(j) = \rho(R_{i-1}, R_i) \leqslant \rho(R'_{i-1}, R'_i)$.

By (2.6), we have

$$\begin{aligned}\sum_{j \in R} \omega(j) &= \sum_{i=1}^{s+1} \sum_{j \in R'_i \setminus R'_{i-1}} \omega(j) \\ &\leqslant \sum_{i=1}^{s+1} (\#R'_i - \#R'_{i-1}) \cdot \rho(R'_{i-1}, R'_i) \\ &= c(R'_{s+1}) - c(R'_0) = c(R).\end{aligned}$$

Therefore, the assertion (iv) holds in this case.

Case 2: $c(R \cup R_s) = n+1$. By Lemma 2.3, and since $\#R \leqslant N+1$, we have

$$\begin{aligned}&\#R \leqslant c(R) + N - n, \text{ and} \\ &n + 1 - c(R_s) = c(R \cup R_s) - c(R_s) \leqslant c(R) - c(R \cap R_s) \leqslant c(R).\end{aligned}$$

Therefore, by the assertion (i), by the definition of Θ and by Lemma 2.3 (ii), applied to R_s and by using (2.3), we have

$$\begin{aligned}\sum_{j\in R}\omega(j) \leqslant \Theta\#R \leqslant \Theta(c(R)+N-n) \\ &= \Theta c(R)\big(1+\frac{N-n}{c(R)}\big) \\ &\leqslant \Theta c(R)\big(1+\frac{N-n}{n+1-c(R_s)}\big) \\ &= c(R)\frac{N+1-c(R_s)}{2N-n+1-\#R_s} \\ &\leqslant c(R).\end{aligned}$$

This completes the proof of Proposition 2.5. □

Definition 2.6. *We call constants $\omega(j)$ $(1 \leqslant j \leqslant q)$ respectively Θ with the properties (i) to (iv) in Proposition 2.5 Nochka weights respectively Nochka constant for hypersurfaces $D_1, \ldots, D_q$ in N-subgeneral position in V, where $N \geq n$ and $q \geq 2N-n+1$.*

Theorem 2.7. *Let $D_1, \ldots, D_q$ be hypersurfaces in N-subgeneral position in V and $\omega(1), \ldots, \omega(q)$ be Nochka weights for them, where $N \geq n$ and $q \geq 2N-n+1$. Consider an arbitrary subset R of $Q := \{1, \ldots, q\}$ with $0 < \#R \leqslant N+1$ and $c^* := c(R)$, and arbitrary nonnegative real constants $E_1, \ldots, E_q$. Then, there exist $j_1, \ldots, j_{c^*} \in R$ such that the hypersurfaces $D_{j_1}, \ldots, D_{j_{c^*}}$ are in general position and*

$$\sum_{j\in R}\omega(j)E_j \leqslant \sum_{i=1}^{c^*} E_{j_i}.$$

Proof. Without loss of the generality, we may assume that $E_1 \geq E_2 \geq \cdots \geq E_q$. We shall choose indices $j_i's$ in R by induction on i. We first choose

$$j_1 := \min\{t \in R\}$$

and set $K_1 := \{k \in R :\ c(\{j_1, k\}) = c(\{j_1\}) = 1\}$. Next, choose

$$j_2 := \min\{t \in R \setminus K_1\}$$

and set $K_2 := \{k \in R :\ c(\{j_1, j_2, k\}) = c(\{j_1, j_2\}) = 2\}$. Similarly, choose

$$j_3 := \min\{t \in R \setminus K_2\}$$

and set $K_3 := \{k \in R: \ c(\{j_1, j_2, j_3, k\}) = c(\{j_1, j_2, j_3\}) = 3\}$. By Lemma 2.2, we can repeat this process until j_{c^*} and K_{c^*}. Then, we have $K_1 \subsetneq K_2 \subsetneq \cdots \subsetneq K_{c^*} = R$. We have $\dim(D_{j_1} \cap \cdots \cap D_{j_i} \cap D_k) = \dim(D_{j_1} \cap \cdots \cap D_{j_i})$, for all $k \in K_i$. Therefore, by the "N-subgeneral position" condition, for any $i \in \{1, \ldots, c^*\}$, there exists an irreducible components σ_i of $D_{j_1} \cap \cdots \cap D_{j_i}$ with $\dim \sigma_i = \dim(D_{j_1} \cap \cdots \cap D_{j_i})$ such that we have that D_k contains σ_i for all $k \in K_i$. Thus, $\dim \cap_{j \in K_i} D_j = \dim(D_{j_1} \cap \cdots \cap D_{j_i}) = n - i$. Then $c(K_i) = i$ for all $i \in \{1, \ldots, c^*\}$.

Set $K_0 := \varnothing$ and $a_i := \sum_{j \in K_i \setminus K_{i-1}} \omega(j), \ i = 1, \ldots, c^*$. Therefore, by Proposition 2.5, we get

$$\sum_{k=1}^{i} a_i = \sum_{j \in K_i} \omega(j) \leqslant c(K_i) = i \ \text{ for all } i \in \{1, \ldots, c^*\}.$$

On the other hand, for any $1 \leqslant i \leqslant c^*$ we have $E_j \leqslant E_{j_i}$ for all $j \in K_i \setminus K_{i-1} (\subseteq R \setminus K_{i-1})$. Thus, we have

$$\begin{aligned}
\sum_{j \in R} \omega(j) E_j &= \sum_{i=1}^{c^*} \sum_{j \in K_i \setminus K_{i-1}} \omega(j) E_j \\
&\leqslant \sum_{i=1}^{c^*} \sum_{j \in K_i \setminus K_{i-1}} \omega(j) E_{j_i} = \sum_{i=1}^{c^*} a_i E_{j_i} \\
&= \sum_{i=1}^{c^*-1} (a_1 + \cdots + a_i)(E_{j_i} - E_{j_{i+1}}) + (a_1 + \cdots + a_{j_{c^*}}) E_{j_{c^*}} \\
&\leqslant \sum_{i=1}^{c^*-1} i (E_{j_i} - E_{j_{i+1}}) + c^* E_{j_{c^*}} \\
&= \sum_{i=1}^{c^*} E_{j_i}.
\end{aligned}$$

This completes the proof of Theorem 2.7. □

3 Some lemmas

Let $X \subset \mathbb{C}P^M$ be a projective variety of dimension n and degree $\triangle$. Let I_X be the prime ideal in $\mathbb{C}[x_0, \ldots, x_M]$ defining X. Denote by $\mathbb{C}[x_0, \ldots, x_M]_m$

the vector space of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_M]$ of degree m (including 0). Put $I_X(m) := \mathbb{C}[x_0, \ldots, x_M]_m \cap I_X$.

The Hilbert function H_X of X is defined by

$$H_X(m) := \dim \mathbb{C}[x_0, \ldots, x_M]_m \diagup I_X(m). \tag{3.1}$$

In particular we have $H_X(m) \leqslant \binom{M+m}{M}$. By the usual theory of Hilbert polynomials, we have

$$H_X(m) := \triangle \cdot \frac{m^n}{n!} + O(m^{n-1}). \tag{3.2}$$

We also need the following result, which should be well known, but since we do not know a good reference, we add a short proof:

Lemma 3.1. *For $n \geq 1$, we have $H_X(m) \geq m + 1$ for all $m \geq 1$.*

Proof. Using the notations introduced above, we first observe that there exists some x_i which is not identically zero on X, without loss of generality we may assume that it is x_0. It suffices to prove the following

CLAIM: For all $m \geq 1$ there exists $i \in \{1, ..., M\}$ such that for all $c_{ij} \in \mathbb{C}$ which are not all zero we have

$$\sum_{j=0}^{m} c_{ij} x_0^{m-j} x_i^j \not\equiv 0 \text{ on } X.$$

In fact, if the claim is true, it means that no (nontrivial) complex linear combination of the $m+1$ monomials $x_0^{m-j} x_i^j$, $j = 0, ..., m$ vanishes identically on X, and, hence, can be contained in $I_X(m)$. So $H_X(m) \geq m + 1$.

Assume that the claim does not hold. Then there exists $m \geq 1$ such that for all $i \in \{1, ..., M\}$ there exist $c_{ij} \in \mathbb{C}$ which are not all zero so that we have

$$\sum_{j=0}^{m} c_{ij} x_0^{m-j} x_i^j \equiv 0 \text{ on } X.$$

Dividing by x_0^m we get that

$$\sum_{j=0}^{m} c_{ij} (\frac{x_i}{x_0})^j \equiv 0 \text{ on } X.$$

This means that the rational functions $\frac{x_i}{x_0}$, $i = 1, ..., M$ on X are all algebraic over $\mathbb{C}$. Since the subset of rational functions on X which are algebraic over $\mathbb{C}$ forms a subfield of the function field $\mathbb{C}(X)$ of X and since (by what we saw above) this subfield contains the rational functions $\frac{x_i}{x_0}$, $i = 1, ..., M$ on X, which generate $\mathbb{C}(X)$ as a field, this means that $\mathbb{C}(X)$ over $\mathbb{C}$ is an algebraic field extension. So the transcendence degree of $\mathbb{C}(X)$ over $\mathbb{C}$ is zero. But by a well know theorem (Hartshorne [11] p.17), observing that we have $\mathbb{C}(X) = \mathbb{C}(X_0)$ and $\dim X = \dim X_0$ if $X_0 = X \cap \{x_0 \neq 0\}$ is one affine chart of X, we get

$$0 = \text{transcendence degree}(\mathbb{C}(X)) = \dim X.$$

With other words, if $n = \dim X \geq 1$, we get a contradiction, proving the claim. □

For each tuple $c = (c_0, \ldots, c_M) \in \mathbb{R}^{M+1}_{\geq 0}$, and $m \in \mathbb{N}$, we define the m-th Hilbert weight $S_X(m, c)$ of X with respect to c by

$$S_X(m, c) := \max \sum_{i=1}^{H_X(m)} I_i \cdot c,$$

where $I_i = (I_{i0}, \ldots, I_{iM}) \in \mathbb{N}_0^{M+1}$ and the maximum is taken over all sets $\{x^{I_i} = x_0^{I_{i0}} \cdots x_M^{I_{iM}}\}$ whose residue classes modulo $I_X(m)$ form a basis of the vector space $\mathbb{C}[x_0, \ldots, x_M]_m \diagup I_X(m)$.

Lemma 3.2. *Let $X \subset \mathbb{C}P^M$ be an algebraic variety of dimension n and degree $\triangle$. Let $m > \triangle$ be an integer and let $c = (c_0, \ldots, c_M) \in \mathbb{R}^{M+1}_{\geq 0}$. Let $\{i_0, \ldots, i_n\}$ be a subset of $\{0, \ldots, M\}$ such that $\{x = (x_0 : \cdots : x_M) \in \mathbb{C}P^M : x_{i_0} = \cdots = x_{i_n} = 0\} \cap X = \varnothing$. Then*

$$\frac{1}{mH_X(m)} S_X(m, c) \geq \frac{1}{(n+1)}(c_{i_0} + \cdots + c_{i_n}) - \frac{(2n+1)\triangle}{m} \cdot \max_{0 \leqslant i \leqslant M} c_i.$$

Proof. We refer to [7], Theorem 4.1, and [8], Lemma 5.1 (or [17], Theorem 2.1 and Lemma 3.2). □

Lemma 3.3 (Theorem 2.3 of [15])**.** *Let f be a linearly nondegenerate holomorphic mapping of $\mathbb{C}$ into $\mathbb{C}P^n$ and let $\{H_j\}_{j=1}^q$ be arbitrary hyperplanes in $\mathbb{C}P^n$. Then for every ϵ,*

$$\Big\| \int_0^{2\pi} \max_{K \in \mathcal{K}} \sum_{j \in K} \log \frac{\|f(re^{i\theta})\| \cdot \|H_j\|}{|H_j(f(re^{i\theta}))|} \frac{d\theta}{2\pi} + N_{W(f)}(r) \leqslant (n+1+\epsilon) T_f(r).$$

where $\mathcal{K}$ is the set of all subsets $K \subset \{1, \ldots, q\}$ such that $\#K = n+1$ and the hyperplanes H_j, $j \in K$ are in general position, $W(f)$ is the Wronskian of f, and $\|H_j\|$ is the maximum of absolute values of the coefficients of H_j.

Lemma 3.4 (Propositions 4.5 and 4.10 of [9]). *Let f be a linearly nondegenerate holomorphic mapping of $\mathbb{C}$ into $\mathbb{C}P^M$ with reduced representation $f = (f_0 : \cdots : f_M)$. Let $W(f) = W(f_0, \ldots, f_M)$ be the Wronskian of f. Then*

$$\nu_{\frac{f_0 \cdots f_M}{W(f)}} \leqslant \sum_{i=0}^{M} \min\{\nu_{f_i}, M\}.$$

4 Proof of Theorem 1.4

We first prove Theorem 1.4 for the case where all the Q_j $(j = 1, ..., q)$ have the same degree d.

Since $D_1, \ldots, D_q$ are in N-subgeneral position in V, we have $\cap_{j=1}^{q} D_j \cap V = \varnothing$. We define a map $\Phi : V \longrightarrow \mathbb{C}P^{q-1}$ by $\Phi(x) = (Q_1(x) : \cdots : Q_q(x))$. Then Φ is a finite morphism (see [19], Theorem 8, page 65). We have that $Y := im\Phi$ is a complex projective subvariety of $\mathbb{C}P^{q-1}$ and $\dim Y = n$ and

$$\triangle := \deg Y \leqslant d^n \cdot \deg V. \tag{4.1}$$

This follows, in the same way as [19], Theorem 8, page 65, from the fact that $\Phi : V \longrightarrow \mathbb{C}P^{q-1}$ is the composition of the restriction of the d-uple embedding $\rho_d|_V : V \longrightarrow \mathbb{C}P^{L-1}$ to V (with $L = \binom{M+d}{M}$) with the linear projection $p : \mathbb{C}P^{L-1} \longrightarrow \mathbb{C}P^{q-1}$, defined by the linear forms $Q_1, ..., Q_q$ in the monomials of degree d, since we have:

$$\deg Y = \deg \Phi(V) \leqslant \deg \rho_d|_V(V) \leqslant d^n \cdot \deg V.$$

It is clear that for any $1 \leqslant i_0 < \cdots < i_n \leqslant q$ such that $\cap_{i=0}^{n} D_{j_i} \cap V = \varnothing$, we have

$$\{y = (y_1 : \cdots : y_q) \in \mathbb{C}P^{q-1} : \ y_{i_0} = \cdots = y_{i_n} = 0\} \cap Y = \varnothing. \tag{4.2}$$

For a positive integer m, denote by $\{I_1, \ldots, I_{q_m}\}$ the set of all $I_i := (I_{i1}, \ldots, I_{iq}) \in \mathbb{N}_0^q$ with $I_{i1} + \cdots + I_{iq} = m$. We have $q_m := \binom{q+m-1}{m}$.

Let F be a holomorphic mapping of $\mathbb{C}$ into $\mathbb{C}P^{q_m-1}$ with the reduced representation $F = \big(Q_1^{I_{11}}(f)\cdots Q_q^{I_{1q}}(f) : \cdots : Q_1^{I_{q_m1}}(f)\cdots Q_q^{I_{q_mq}}(f)\big)$, (note that $Q_1^m(f), \dots, Q_q^m(f)$ have no common zero point).

Define an isomorphism between vector spaces, $\Psi : \mathbb{C}[z_1, \dots, z_{q_m}]_1 \longrightarrow \mathbb{C}[y_1, \dots, y_q]_m$ by $\Psi(z_i) := y^{I_i} \quad (i = 1, \dots, q_m)$. Consider the vector space $\mathcal{H} := \{H \in \mathbb{C}[z_1, \dots, z_{q_m}]_1 : H(F) \equiv 0\ \}$. Then F is a linearly nondegenerate mapping of $\mathbb{C}$ into the complex projective space $P := \cap_{H\in\mathcal{H}}\{H = 0\} \subset \mathbb{C}P^{q_m-1}$, and we will from now on, by abuse of notation, consider F to be this linearly nondegenerate map $F : \mathbb{C} \to P$.

For any linear form $H \in \mathbb{C}[z_1, \dots, z_{q_m}]_1$, since f is algebraically nondegenerate, we have that $H \in \mathcal{H}$ if only if

$$H(Q_1^{I_{11}}(x)\cdots Q_q^{I_{1q}}(x), \cdots, Q_1^{I_{q_m1}}(x)\cdots Q_q^{I_{q_mq}}(x)) \equiv 0 \ \textit{on}\ V.$$

This is possible if and only if $\Psi(H)(y) := H(y^{I_1}, \cdots, y^{I_{q_m}}) \equiv 0$ on Y. Therefore, we get that $\Psi(\mathcal{H}) = (I_Y)_m$. On the other hand Ψ is an isomorphism. Hence, we have

$$\begin{aligned}\dim P = \dim \bigcap_{H\in\mathcal{H}}\{H = 0\} &= q_m - 1 - \dim \mathcal{H}\\ &= q_m - 1 - \dim(I_Y)_m = H_Y(m) - 1. \qquad (4.3)\end{aligned}$$

We define hyperplanes H_j $(j = 1, \dots, q_m)$ in the complex projective space P by $H_j := \{(z_1 : \cdots : z_{q_m}) \in \mathbb{C}P^{q_m-1} :\ z_j = 0\} \cap P$, (these intersections are not empty by Bézout's theorem, and they are proper algebraic subsets of P since $V \not\subset D_k,\ 1 \leqslant k \leqslant q$).

Denote by $\mathcal{L}$ the set of all subsets J of $\{1, \dots, q_m\}$ such that $\#J = H_Y(m)$ and the hyperplanes $H_j, j \in J$, are in general position in P. Since Ψ is an isomorphism and $\Psi(\mathcal{H}) = I_Y(m)$, $\mathcal{L}$ is also the set of all subsets J of $\{1, \dots, q_m\}$ such that $\{y^{I_j},\ j \in J\}$ is a basis of $\mathbb{C}[y_1, \dots, y_q]_m \big/ I_Y(m)$.

For each $j \in \{1, \dots, q\}$ and $k \in \{1, \dots, q_m\}$, we put

$$E_{D_j}(f) = \log \frac{\|f\|^d \cdot \|Q_j\|}{|Q_j(f)|} \geq 0 \ \text{ and } \ E_{H_k}(F) = \log \frac{\|F\| \cdot \|H_k\|}{|H_k(F)|} \geq 0,$$

where $\|Q_j\|$ (respectively $\|H_k\|$) is the maximum of absolute values of the coefficients of Q_j (respectively H_k). They are continuous functions with values in $\mathbb{R}_{\geq 0} \cup \{+\infty\}$ which take the value $+\infty$ only on discrete subsets of $\mathbb{C}$.

Denote by $\mathcal{K}$ the set of all subsets K of $\{1, \ldots, q\}$ such that $\#K = n+1$ and $\cap_{j\in K} D_j \cap V = \varnothing$. Let $\mathcal{N}$ be the set of all subsets $J \subset \{1, \ldots, q\}$ with $\#J = N+1$. Let $\{\omega(j)\}_{j=1}^q$ and Θ be Nochka weights and Nochka constant for the hypersurfaces D_j in N-subgeneral position in V. By Theorem 2.7, for any $z \in \mathbb{C}$ and any $J \in \mathcal{N}$, there exists a subset $K(J,z) \in \mathcal{K}$, such that

$$\sum_{j\in J} \omega(j) E_{D_j}(f(z)) \leqslant \sum_{j\in K(J,z)} E_{D_j}(f(z)). \tag{4.4}$$

For any $J \in \mathcal{N}$, since the hypersurfaces D_j $(j = 1, \ldots, q)$ are in N-subgeneral position in V, the function $\lambda_J(x) := \frac{\max_{j\in J} |Q(x)|}{\|x\|^d}$ is continuous on V and $\lambda_J(x) > 0$ for all $x \in V$. On the other hand, V is compact, so there exist positive constants c_J, c'_J such that $c'_J \geq \lambda_J(f(z)) \geq c_J$ for all $z \in \mathbb{C}$. This implies that

$$d \cdot \log \|f\| = \max_{j\in J} \log |Q(f)| + O(1), \text{ for all } J \in \mathcal{N}. \tag{4.5}$$

Therefore, there exists a positive constant c such that

$$\min_{\{j_1,\ldots,j_{q-N-1}\}} \sum_{i=1}^{q-N-1} E_{D_{j_i}}(f) \leqslant c.$$

Then, we have

$$\sum_{j=1}^{q} \omega(j) E_{D_j}(f) \leqslant \max_{J\in\mathcal{N}} \sum_{j\in J} \omega(j) E_{D_j}(f) + O(1). \tag{4.6}$$

By (4.4) and (4.6), for every $z \in \mathbb{C}$, we have

$$\begin{aligned}\sum_{j=1}^{q} \omega(j) E_{D_j}(f(z)) &\leqslant \max_{J\in\mathcal{N}} \sum_{j\in K(J,z)} E_{D_j}(f(z)) + O(1) \\ &\leqslant \max_{K\in\mathcal{K}} \sum_{j\in K} E_{D_j}(f(z)) + O(1).\end{aligned}$$

This implies that

$$\begin{aligned}\sum_{j=1}^{q} \omega(j) d \log \|f\| - \sum_{j=1}^{q} \omega(j) \log |Q_j(f)| &\leqslant \sum_{j=1}^{q} \omega(j) E_{D_j}(f) + O(1) \\ &\leqslant \max_{K\in\mathcal{K}} \sum_{j\in K} E_{D_j}(f) + O(1). \end{aligned} \tag{4.7}$$

Applying integration on the both sides of (4.7), using Proposition 2.5 and Jensen's formula, we get

$$
\begin{aligned}
d\big(\Theta(q-2N+n-1)+n+1\big)T_f(r) - \sum_{j=1}^{q}\omega(j)N_f(r,D_j)& \\
\leqslant \int_0^{2\pi}\max_{K\in\mathcal{K}}\sum_{j\in K}E_{D_j}(f(re^{i\theta}))\frac{d\theta}{2\pi}+O(1).& \qquad (4.8)
\end{aligned}
$$

Since $ImF \subset P$ and $\{Q_1^{I_{i1}}(f)\cdots Q_q^{I_{iq}}(f),\ 1\leqslant i\leqslant q_m\}$ have no common zero point, for every $J\in\mathcal{L}$, the holomorphic functions $\{Q_1^{I_{i1}}(f)\cdots Q_q^{I_{iq}}(f),\ i\in J\}$ also have no common zero point.
Then, for every $J\in\mathcal{L}$, we have

$$
\begin{aligned}
\|F\| = \max_{i\in J}|H_i(F)|+O(1) &= \max_{i\in J}|Q_1^{I_{i1}}(f)\cdots Q_q^{I_{iq}}(f)|+O(1)\\
&\leqslant \|f\|^{dm}+O(1).
\end{aligned}
$$

This implies that

$$
T_F(r)\leqslant dm\cdot T_f(r)+O(1). \qquad (4.9)
$$

For every $J\in\mathcal{L}$ and $i\in J$, we have

$$
\begin{aligned}
E_{H_i}(F) &= \log\frac{\|F\|\cdot\|H_i\|}{|\ H_i(F)\ |} = \log\frac{\|F\|}{|Q_1^{I_{i1}}(f)\cdots Q_q^{Iiq}(f)|}+O(1)\\
&= \log\frac{\|f\|^{dm}}{|Q_1^{I_{i1}}(f)\cdots Q_q^{Iiq}(f)|} - dm\log\|f\|+\log\|F\|+O(1)\\
&= \sum_{1\leqslant j\leqslant q} I_{ij}E_{D_j}(f) - dm\log\|f\|+\log\|F\|+O(1). \qquad (4.10)
\end{aligned}
$$

Let $c_z := (E_{D_1}(f(z)),\cdots,E_{D_q}(f(z)))$ for every $z\in\mathbb{C}\setminus D$, where D denotes the discrete subset where one of these functions takes the value $+\infty$. By the definition of the Hilbert weight, there exists a subset $J_z\in\mathcal{L}$ such that

$$
S_Y(m,c_z)=\sum_{i\in J_z} I_i\cdot c_z. \qquad (4.11)
$$

By (4.2) and by Lemma 3.2, for every $m > \triangle$ and $K \in \mathcal{K}$, we have

$$\frac{S_Y(m, c_z)}{mH_Y(m)} \geq \frac{1}{n+1} \sum_{j \in K} E_{D_j}(f(z)) - \frac{(2n+1)\triangle}{m} \max_{1 \leqslant j \leqslant q} E_{D_j}(f(z)). \tag{4.12}$$

Then, by (4.10), (4.11) and (4.12), for every $K \in \mathcal{K},\ z \in \mathbb{C} \setminus D$, we have

$$\begin{aligned}
\frac{1}{(n+1)} \sum_{j \in K} E_{D_j}(f(z)) &\leqslant \frac{S_Y(m, c_z)}{mH_Y(m)} + \frac{(2n+1)\triangle}{m} \max_{1 \leqslant j \leqslant q} E_{D_j}(f(z)) \\
&= \frac{\sum_{i \in J_z} I_i \cdot c_z}{mH_Y(m)} + \frac{(2n+1)\triangle}{m} \max_{1 \leqslant j \leqslant q} E_{D_j}(f(z)) \\
&= \frac{1}{mH_Y(m)} \sum_{\substack{i \in J_z \\ 1 \leqslant j \leqslant q}} I_{ij} E_{D_j}(z) + \frac{(2n+1)\triangle}{m} \max_{1 \leqslant j \leqslant q} E_{D_j}(f(z)) \\
&= \frac{1}{mH_Y(m)} \sum_{i \in J_z} E_{H_i}(F(z)) + d \log \|f(z)\| - \frac{1}{m} \log \|F(z)\| \\
&\qquad + \frac{(2n+1)\triangle}{m} \max_{1 \leqslant j \leqslant q} E_{D_j}(f(z)) + O(1) \\
&\leqslant \frac{1}{mH_Y(m)} \max_{L \in \mathcal{L}} \sum_{i \in L} E_{H_i}(F(z)) + d \log \|f(z)\| - \frac{1}{m} \log \|F(z)\| \\
&\qquad + \frac{(2n+1)\triangle}{m} \sum_{1 \leqslant j \leqslant q} E_{D_j}(f(z)) + O(1).
\end{aligned} \tag{4.13}$$

This implies that, for every $z \in \mathbb{C} \setminus D$,

$$\begin{aligned}
\max_{K \in \mathcal{K}} \frac{1}{(n+1)} \sum_{j \in K} E_{D_j}(f(z)) \leqslant \frac{1}{mH_Y(m)} & \max_{L \in \mathcal{L}} \sum_{i \in L} E_{H_i}(F(z)) + d \log \|f(z)\| \\
& - \frac{1}{m} \log \|F(z)\| + \frac{(2n+1)\triangle}{m} \sum_{1 \leqslant j \leqslant q} E_{D_j}(z) + O(1),
\end{aligned}$$

and by continuity this then holds for all $z \in \mathbb{C}$. So, by integrating and by

(4.8), we get

$$d\big(\Theta(q-2N+n-1)+n+1\big)T_f(r)-\sum_{j=1}^{q}\omega(j)N_f(r,D_j)$$
$$\leqslant \frac{n+1}{mH_Y(m)}\int_0^{2\pi}\max_{L\in\mathcal{L}}\sum_{i\in L}E_{H_i}(F(re^{i\theta}))\frac{d\theta}{2\pi}+d(n+1)T_f(r)-\frac{n+1}{m}T_F(r)$$
$$+\frac{(2n+1)(n+1)\triangle}{m}\sum_{1\leqslant j\leqslant q}\int_0^{2\pi}E_{D_j}(re^{i\theta})\frac{d\theta}{2\pi}+O(1). \qquad (4.14)$$

By (4.3) and Lemma 3.3 (with $\epsilon=1$), we have

$$\Big\|\frac{n+1}{mH_Y(m)}\int_0^{2\pi}\max_{L\in\mathcal{L}}\sum_{i\in L}E_{H_i}(F(re^{i\theta}))\frac{d\theta}{2\pi}$$
$$\leqslant \frac{(n+1)(H_Y(m)+1)}{mH_Y(m)}T_F(r)-\frac{n+1}{mH_Y(m)}N_{W(F)}(r). \qquad (4.15)$$

For each $j\in\{1,\ldots,q\}$, by Jensen's formula, we have

$$\int_0^{2\pi}E_{D_j}(re^{i\theta})\frac{d\theta}{2\pi}\leqslant d\int_0^{2\pi}\log\|f(re^{i\theta})\|\frac{d\theta}{2\pi}-\int_0^{2\pi}\log|Q_j(re^{i\theta})|\frac{d\theta}{2\pi}+O(1)$$
$$\leqslant dT_f(r)-N_f(r,D_j)+O(1)\leqslant dT_f(r)+O(1). \qquad (4.16)$$

For an arbitrary $\epsilon>0$, we choose

$$m:=[4d^{n+1}q(2n+1)(2N-n+1)\deg V\cdot\frac{1}{\epsilon}]+1\,.$$

Then, assuming without loss of generality that $\epsilon\leqslant 1$, by (4.1), by Lemma 3.1 and by Proposition 2.5 (iii) we have $m>\triangle$, which we assumed for (4.12), and

$$\frac{(2n+1)(n+1)dq\triangle}{m}<\frac{\Theta\epsilon}{4}\text{ and }\frac{(n+1)d}{H_Y(m)}<\frac{\Theta\epsilon}{4}. \qquad (4.17)$$

Then, by (4.9), (4.14), (4.15), and (4.16), we get

$$\Big\|\big(\Theta(q-2N+n-1)+n+1\big)dT_f(r)-\sum_{j=1}^{q}\omega(j)N_f(r,D_j)$$
$$\leqslant\big((n+1)d+\frac{\Theta\epsilon}{2}\big)T_f(r)-\frac{n+1}{mH_Y(m)}N_{W(F)}(r).$$

Therefore,

$$\Big\|\Theta d(q-2N+n-1-\frac{\epsilon}{2})T_f(r)\leqslant\sum_{j=1}^{q}\omega(j)N_f(r,D_j)-\frac{n+1}{mH_Y(m)}N_{W(F)}(r) \tag{4.18}$$

For each $J:=\{j_1,\dots,j_{H_Y(m)}\}\in\mathcal{L}$, then there exists a constant $\gamma_J\in\mathbb{C},\gamma_J\neq 0$ such that

$$W(F)=\gamma_J\cdot W(Q_1^{I_{j_1 1}}(f)\cdots Q_q^{I_{j_1 q}}(f),\dots,Q_1^{I_{j_{H_Y(m)}1}}(f)\cdots Q_q^{I_{j_{H_Y(m)}q}}(f)).$$

On the other hand, by (4.3) and Lemma 3.4,

$$\nu_{\frac{Q_1^{I_{j_1 1}}(f)\cdots Q_q^{I_{j_1 q}}(f)\cdots Q_1^{I_{j_{H_Y(m)}1}}(f)\cdots Q_q^{I_{j_{H_Y(m)}q}}(f)}{W\left(Q_1^{I_{j_1 1}}(f)\cdots Q_q^{I_{j_1 q}}(f),\dots,Q_1^{I_{j_{H_Y(m)}1}}(f)\cdots Q_q^{I_{j_{H_Y(m)}q}}(f)\right)}}\leqslant\sum_{1\leqslant i\leqslant H_Y(m)}\nu^{[H_Y(m)-1]}_{Q_1^{I_{j_i 1}}(f)\cdots Q_q^{I_{j_i q}}(f)}.$$

Hence, for all $J\in\mathcal{L}$, we have

$$\begin{aligned}\nu_{W(F)}&\geq\nu_{Q_1^{I_{j_1 1}}(f)\cdots Q_q^{I_{j_1 q}}(f)\cdots Q_1^{I_{j_{H_Y(m)}1}}(f)\cdots Q_q^{I_{j_{H_Y(m)}q}}(f)}\\&\quad-\sum_{1\leqslant i\leqslant H_Y(m)}\nu^{[H_Y(m)-1]}_{Q_1^{I_{j_i 1}}(f)\cdots Q_q^{I_{j_i q}}(f)}\\&\geq\sum_{1\leqslant j\leqslant q}\sum_{i\in J}I_{ij}\big(\nu_{Q_j(f)}-\nu^{[H_Y(m)-1]}_{Q_j(f)}\big).\end{aligned}\tag{4.19}$$

For every $z\in\mathbb{C}$, let $c_z:=(c_{1,z},\dots,c_{q,z})$ where $c_{j,z}:=\nu_{Q_j(f)}(z)-\nu^{[H_Y(m)-1]}_{Q_j(f)}(z)$. Then, by definition of the Hilbert weight, there exists $J_z\in\mathcal{L}$ such that

$$S_Y(m,c_z)=\sum_{i\in J_z}I_i\cdot c_z=\sum_{1\leqslant j\leqslant q}\sum_{i\in J_z}I_{ij}\big(\nu_{Q_j(f)}(z)-\nu^{[H_Y(m)-1]}_{Q_j(f)}(z)\big).$$

Then, by (4.2) and Lemma 3.2, for every $K \in \mathcal{K}$ we have

$$\begin{aligned}
\frac{1}{mH_Y(m)} \sum_{1 \leqslant j \leqslant q} \sum_{i \in J_z} I_{ij} \big(\nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z)\big) \\
\geq \frac{1}{n+1} \sum_{j \in K} \big(\nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z)\big) \\
- \frac{(2n+1)\triangle}{m} \max_{1 \leqslant j \leqslant q} \big(\nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z)\big) \\
\geq \frac{1}{n+1} \sum_{j \in K} \big(\nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z)\big) \\
- \frac{(2n+1)\triangle}{m} \sum_{1 \leqslant j \leqslant q} \nu_{Q_j(f)}(z).
\end{aligned}$$

Combining with (4.19), for every $K \in \mathcal{K}$ and $z \in \mathbb{C}$, we have

$$\begin{aligned}
\frac{1}{mH_Y(m)} \nu_{W(F)(z)} \geq \frac{1}{n+1} \sum_{j \in K} \big(\nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z)\big) \\
- \frac{(2n+1)\triangle}{m} \sum_{1 \leqslant j \leqslant q} \nu_{Q_j(f)}(z).
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{n+1}{mH_Y(m)} \nu_{W(F)} \geq \max_{K \in \mathcal{K}} \sum_{j \in K} \big(\nu_{Q_j(f)} - \nu^{[H_Y(m)-1]}_{Q_j(f)}\big) \\
- \frac{(n+1)(2n+1)\triangle}{m} \sum_{1 \leqslant j \leqslant q} \nu_{Q_j(f)}. \quad (4.20)
\end{aligned}$$

By Theorem 2.7, for any $z \in \mathbb{C}$ and any $J \in \mathcal{N}$, there exists subset $K'(J, z) \in \mathcal{K}$, such that

$$\begin{aligned}
\sum_{j \in J} \omega(j) \big(\nu_{Q_j(f(z))} - \nu^{[H_Y(m)-1]}_{Q_j(f(z))}\big) \leqslant \sum_{j \in K'(J,z)} \big(\nu_{Q_j(f(z))} - \nu^{[H_Y(m)-1]}_{Q_j(f(z))}\big) \\
\leqslant \max_{K \in \mathcal{K}} \sum_{j \in K} \big(\nu_{Q_j(f(z))} - \nu^{[H_Y(m)-1]}_{Q_j(f(z))}\big).
\end{aligned}$$

This implies that

$$\max_{J\in\mathcal{N}}\sum_{j\in J}\omega(j)\big(\nu_{Q_j(f)}-\nu^{[H_Y(m)-1]}_{Q_j(f)}\big)\leqslant\max_{K\in\mathcal{K}}\sum_{j\in K}\big(\nu_{Q_j(f)}-\nu^{[H_Y(m)-1]}_{Q_j(f)}\big). \quad (4.21)$$

On the other hand, since the hypersurfaces D_j $(j=1,\dots,q)$ are in N-subgeneral position in V, we have that for any $z\in\mathbb{C}$ there are at least (q-N) indices j of $\{1,\dots,q\}$ such that $\nu_{Q_j(f)}(z)=0$. Thus, we have

$$\sum_{j=1}^{q}\omega(j)\big(\nu_{Q_j(f)}-\nu^{[H_Y(m)-1]}_{Q_j(f)}\big)=\max_{J\in\mathcal{N}}\sum_{j\in J}\omega(j)\big(\nu_{Q_j(f)}-\nu^{[H_Y(m)-1]}_{Q_j(f)}\big).$$

Combining with (4.21), we have

$$\sum_{j=1}^{q}\omega(j)\big(\nu_{Q_j(f)}-\nu^{[H_Y(m)-1]}_{Q_j(f)}\big)\leqslant\max_{K\in\mathcal{K}}\sum_{j\in K}\big(\nu_{Q_j(f)}-\nu^{[H_Y(m)-1]}_{Q_j(f)}\big).$$

Therefore, by (4.20) we have

$$\frac{n+1}{mH_Y(m)}\nu_{W(F)}\geq\sum_{j=1}^{q}\omega(j)\big(\nu_{Q_j(f)}-\nu^{[H_Y(m)-1]}_{Q_j(f)}\big)-\frac{(n+1)(2n+1)\triangle}{m}\sum_{1\leqslant j\leqslant q}\nu_{Q_j(f)}.$$

So, by integrating and by Jensen's formula, we get

$$\begin{aligned}\frac{n+1}{mH_Y(m)}N_{W(F)}(r)&\geq\sum_{j=1}^{q}\omega(j)\big(N_f(r,D_j)-N_f^{[H_Y(m)-1]}(r,D_j)\big)\\&\quad-\frac{(n+1)(2n+1)\triangle}{m}\sum_{1\leqslant j\leqslant q}N_f(r,D_j)\\&\geq\sum_{j=1}^{q}\omega(j)\big(N_f(r,D_j)-N_f^{[H_Y(m)-1]}(r,D_j)\big)\\&\quad-\frac{(n+1)(2n+1)dq\triangle}{m}\sum_{1\leqslant j\leqslant q}T_f(r)-O(1)\\&\geq\sum_{j=1}^{q}\omega(j)\big(N_f(r,D_j)-N_f^{[H_Y(m)-1]}(r,D_j)\big)-\frac{\Theta\epsilon}{4}T_f(r).\end{aligned}$$

Combining with (4.18) we get

$$\Big\|\Theta d(q-2N+n-1-\epsilon)T_f(r)\leqslant\sum_{j=1}^{q}\omega(j)N_f^{[H_Y(m)-1]}(r,D_j).$$

On the other hand, $\omega(j)\leqslant\Theta$ by Proposition 2.5 (i), therefore

$$\Big\|(q-2N+n-1-\epsilon)T_f(r)\leqslant\sum_{j=1}^{q}\frac{1}{d}N_f^{[H_Y(m)-1]}(r,D_j). \tag{4.22}$$

This completes the proof of Theorem 1.4 and Proposition 1.5 in the special case of $\deg Q_j=d$ by the fact that $H_Y(m)-1\leqslant\binom{q+m-1}{m}-1$, note that $Y\subset\mathbb{C}P^{q-1}$.

We now prove the theorem for the general case: $\deg Q_j=d_j$. Denote by d the least common multiple of $d_1,\dots,d_q$ and put $d_j^*:=\frac{d}{d_j}$. By (4.22) with the hypersurfaces $Q_j^{d_j^*}$ $(j\in\{1\dots,q\})$ of common degree d, we have

$$\begin{aligned}\|(q-2N+n-1-\varepsilon)T_f(r)&\leqslant\sum_{j=1}^{q}\frac{1}{d}N_f^{[H_Y(m)-1]}(r,Q_j^{d_j^*})\\&\leqslant\sum_{j=1}^{q}\frac{d_j^*}{d}N_f^{[[\frac{H_Y(m)-1}{d_j^*}+1]]}(r,Q_j)\\&\leqslant\sum_{j=1}^{q}\frac{1}{d_j}N_f^{[L_j]}(r,Q_j),\end{aligned}$$

where

$$L_j:=[\frac{d_j(H_Y(m)-1)}{d}+1]\leqslant\Big[\frac{d_j\binom{q+m-1}{m}}{d}+1\Big].$$

This completes the proof of Theorem 1.4 and of Proposition 1.5. □

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