Examples of plane curves of low degrees with hyperbolic and C–hyperbolic complements

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Abstract

We find explicit examples of families of irreducible plane projective curves $C \subset \mathbb{P}^2$ of any even degree starting with degree six, such that the complement $\mathbb{P}^2 \setminus C$ is C–hyperbolic. The latter means that some covering over $\mathbb{P}^2 \setminus C$ is Carathéodory hyperbolic. This implies that the complement $\mathbb{P}^2 \setminus C$ is Kobayashi hyperbolic and (due to Lin's Theorem) that the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ does not contain a nilpotent subgroup of finite index.

1 Introduction

A complex space X is said to be *C-hyperbolic* if there exists a non-ramified covering $Y \to X$ such that Y is Carathéodory hyperbolic, i.e. the points in Y are separated by bounded holomorphic functions [Ko]. If there exists a covering Y of X such that the sets of constancy of all the bounded holomorphic functions on Y are at most finite, we say that X is *almost C-hyperbolic.* Remind the general problem: *Which quasiprojective varieties are uniformized by bounded domains in* $\mathbb{C}^n$*?* In particular, such a variety has to be C-hyperbolic. Here we study plane projective curves whose complements are C-hyperbolic. We prove the following

1.1. Theorem. *Let* $C \subset \mathbb{P}^2$ *be an irreducible curve of genus* g *and* C^* *its dual curve. Let* $n = \deg C^*$. *Assume that* C^* *is an immersed curve.*
a) If $g \geq 1$, *then* $\mathbb{P}^2 \setminus C$ *is C–hyperbolic.*
b) If $g = 0$ *and* C^* *is a generic rational nodal curve of degree* $n \geq 5$, *then* $\mathbb{P}^2 \setminus C$ *is almost C–hyperbolic.*

c) In both cases $\mathbb{P}^2 \setminus C$ is Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$.

Consider, for instance, the elliptic sextics with 9 cusps (see (4.2)). Such a sextic is dual to a smooth cubic and hence, due to (a) it has C–hyperbolic complement. Moreover, it can be given explicitly by Schläfli's equation. Actually, 6 is the least possible degree of an irreducible plane curve with C–hyperbolic complement (see (3.4)).

Note that C–hyperbolicity is a much stronger property than Kobayashi hyperbolicity. Whereas hyperbolicity of projective complements is often stable (see e.g. [Za2]), C–hyperbolicity can be easily destroyed by small deformations (we clarify this later on in this introduction).

S. Kobayashi [Ko] proposed the following

1.2. Conjecture. *Let $\mathcal{H}(d)$ be the set of all hypersurfaces D of degree d in $\mathbb{P}^n$ such that $\mathbb{P}^n \setminus D$ is complete hyperbolic and hyperbolically embedded into $\mathbb{P}^n$. Then for any $d \geq 2n+1$ the set $\mathcal{H}(d)$ contains a Zariski open subset of $\mathbb{P}^{N(d)}$, where $\mathbb{P}^{N(d)}$ is the complete linear system of effective divisors of degree d in $\mathbb{P}^n$.*

While for $n > 2$ the problem is still open, for $n = 2$ Y.-T. Siu and S.-K. Yeung [SY] have announced a proof of Conjecture 1.2 for plane curves of sufficiently large degree ($d > 10^6$). However, even for $n = 2$ and for small d it is not so easy to construct explicit examples of (families of) irreducible plane curves in $\mathcal{H}(d)$ (see [Za3] and literature therein). For reducible curves this is easier, cf. e.g. [DSW1,2].

The first examples of smooth curves in $\mathcal{H}(d)$ of any even degree $d \geq 30$ were constructed by K. Azukawa and M. Suzuki [AZ]. A. Nadel [Nad] mentioned such examples for any $d \geq 18$ which is divisible by 6. K. Masuda and J. Noguchi [MN] obtained smooth curves in $\mathcal{H}(d)$ for any $d \geq 21$.

In [Za2] the existence of smooth curves in $\mathcal{H}(d)$ is proven (by deformation arguments) for arbitrary $d \geq 5$; however, their equations are non–explicit. For instance, the equations of smooth quintics in $\mathcal{H}(5)$ include five parameters which should be chosen successively small enough, with unexplicit upper bounds.

In a series of papers by M. Green [Gr1], J. Carlson and M. Green [CaGr] and H. Grauert and U. Peternell [GP] sufficient conditions were found for irreducible plane curves of genus $g \geq 2$ to be in $\mathcal{H}(d)$. This leads to examples of irreducible (but singular) curves in $\mathcal{H}(d)$ with $d \geq 9$ (cf. section 4).

Generalizing their methods in the proof of Theorem 1.1, for any even $d \geq 6$ we obtain families of irreducible curves in $\mathcal{H}(d)$ described in terms of genus and singularities. While all the examples known before were curves of genus at least two, we get examples of elliptic or rational such curves. They all have at least classical singularities, and the method used is not available to get smooth such examples. On the other hand, it is clear that an elliptic or rational curve with hyperbolic complement must be singular.

One of our examples of curves with hyperbolically embedded complements in degree 6 is the family of rational sextics with six cusps at a conic and four nodes (4.5). Such a curve is dual to a generic rational nodal quartic, and therefore, one can easily write down its explicit equation.

A complex space Y is called *Liouville* if all the bounded holomorphic functions on Y are constant. This property is just opposite of being Carathéodory hyperbolic. By Lin's Theorem [Li, Theorem B], a Galois covering Y of a quasiprojective variety X is Liouville if its group of deck transformations is *almost nilpotent*, i.e. contains a nilpotent subgroup of finite index. It follows that any covering over $\mathbb{P}^2 \setminus C$ is a Liouville one as soon as the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is almost nilpotent. In particular, this is so for a nodal (not necessarily irreducible) plane curve C. Indeed, due to the Deligne-Fulton Theorem [De, Fu], in the latter case the group $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian. As a corollary we obtain that for curves mentioned in Theorem 1.1 the group $\pi_1(\mathbb{P}^2 \setminus C)$ is not almost nilpotent (cf. (3.3)).

The same arguments show that C–hyperbolicity of $\mathbb{P}^2 \setminus C$ can be easily destroyed under small deformations of C. Indeed, one loses C–hyperbolicity when passing to a smooth or nodal curve C' of the same degree as C which approximates C.

The paper is organized as follows. In section 2 we summarize the necessary background on plane algebraic curves, hyperbolic complex analysis and on $\mathbb{P}GL(2,\mathbb{C})$–actions on $\mathbb{P}^n$. In section 3 we discuss Theorem 1.1 and its generalization. In section

4 we look at families of curves of low degrees with hyperbolic and C–hyperbolic complements. In particular, we obtain examples of irreducible plane curves such that the fundamental group of the complement is not almost nilpotent.

2 Preliminaries

a) Background on plane algebraic curves

We say that a curve C in $\mathbb{P}^2$ has *classical singularities* if its singular points are nodes and ordinary cusps. It is called *a Plücker curve* if both C and the dual curve C^* have classical singularities only and no flex at a node. If the normalization mapping $\nu : C^*_{norm} \to C \hookrightarrow \mathbb{P}^2$ is an immersion, or, which is equivalent, if all irreducible local analytic branches of C are smooth, then we say that C is *an immersed curve.*

Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree $d \geq 2$ and of geometric genus g. Then $d^* = \deg C^*$ (i.e. the class of C) is defined by the class formula (see [Na, (1.5.4)])

$$d^* = 2(g + d - 1) - \sum_{p \in \mathrm{sing}\, C} (m_p - r_p) \tag{1}$$

where $m_p = \mathrm{mult}_p C$, r_p is the number of irreducible analytic branches of C at p.

We will need the following corollary of the genus formula (see [Na, (2.1.10)]):

$$2g \leq (d-1)(d-2) - \sum_{p \in \mathrm{sing}\, C} m_p(m_p - 1)$$

and $2g = (d-1)(d-2) - 2\delta$ for a nodal curve with δ nodes. For reader's convenience we recall here also the usual Plücker formulas:

$$g = 1/2(d-1)(d-2) - \delta - \kappa = 1/2(d^*-1)(d^*-2) - b - f$$

$$d^* = d(d-1) - 2\delta - 3\kappa \quad \text{and} \quad d = d^*(d^*-1) - 2b - 3f$$

for a Plücker curve C with δ nodes, κ cusps, b bitangent lines and f flexes.

Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree $d \geq 2$ and let $\nu : C^*_{norm} \to C^*$ be the normalization of the dual curve. Following Zariski [Zar, p.307, p.326] and

M. Green [Gr1] (see also [DL]), we consider the mapping $\rho_C : \mathbb{P}^2 \to S^n C^*_{norm}$ of $\mathbb{P}^2$ into the n-th symmetric power of C^*_{norm}, where $n = \deg C^*$. We put $\rho_C(z) = \nu^*(l_z) \subset S^n C^*_{norm}$, where $z \in \mathbb{P}^2$ and $l_z \subset \mathbb{P}^{2*}$ is the dual line. It is easy to check that $\rho_C : \mathbb{P}^2 \to S^n C^*_{norm}$ is a holomorphic embedding, which we call in the sequel *the Zariski embedding.* We denote by D_n the union of the diagonal divisors in $(C^*_{norm})^n$ and by $R_n = s_n(D_n)$ the discriminant divisor, i.e. the ramification locus of the branched covering $s_n : (C^*_{norm})^n \to S^n C^*_{norm}$. Thus, we have the diagram

$$\begin{array}{ccc} & & (C^*_{norm})^n \supset D_n \\ & & s_n \downarrow \qquad \downarrow \\ C \subset \mathbb{P}^2 & \overset{\rho_C}{\hookrightarrow} & S^n(C^*_{norm}) \supset R_n \end{array} \qquad (2)$$

b) On the Vieta map and the $\mathbb{P}GL(2,\mathbb{C})$–action on $\mathbb{P}^n$

The symmetric power $S^n\mathbb{P}^1$ can be identified with $\mathbb{P}^n$ in such a way that the canonical projection $s_n : (\mathbb{P}^1)^n \to S^n\mathbb{P}^1$ becomes *the Vieta ramified covering*, which is given by

$$((u_1 : v_1), \ldots, (u_n : v_n)) \longmapsto$$
$$\longmapsto (\prod_{i=1}^n v_i)\,(1 : \sigma_1(u_1/v_1, \ldots, u_n/v_n) \;:\; \ldots \;:\; \sigma_n(u_1/v_1, \ldots, u_n/v_n)) \;,$$

where $\sigma_i(x_1, \ldots, x_n)\,,\; i = 1, \ldots, n$, are the elementary symmetric polynomials. This is a Galois covering with the Galois group being the n-th symmetric group S_n. With $z_i := (u_i : v_i) \in \mathbb{P}^1,\; i = 1, \ldots, n$, we have $s_n(z_1, \ldots, z_n) = (a_0 : \ldots : a_n)$, where $z_i,\; i = 1, \ldots, n,$ are the roots of the binary form $\sum\limits_{i=0}^n a_i u^{n-i} v^i$ of degree n, see [Zar, p.252].

Note that the Vieta map $s_n : (\mathbb{P}^1)^n \to S^n\mathbb{P}^1 = \mathbb{P}^n$ is equivariant with respect to the natural actions of the group $\mathbb{P}GL(2,\mathbb{C}) = \mathrm{Aut}\,\mathbb{P}^1$ on $(\mathbb{P}^1)^n$ and on $\mathbb{P}^n$, respectively. The branching divisors D_n (the union of the diagonals) resp. R_n (the discriminant divisor), as well as their complements are invariant under the corresponding actions. It is easily seen that for $n \geq 3$ the orbit space of the $\mathbb{P}GL(2,\mathbb{C})$–action on $\mathbb{P}^n \setminus R_n$ is naturally isomorphic to the moduli space $M_{0,n}$ of the Riemann sphere with n punctures. Denote by $\tilde{M}_{0,n}$ the quotient $((\mathbb{P}^1)^n \setminus D_n)\,/\mathbb{P}GL(2,\mathbb{C})$. We have the following commutative diagram of equivariant morphisms

$$
\begin{array}{ccc}
(\mathbb{P}^1)^n \setminus D_n & \xrightarrow{\tilde{\pi}_n} & \tilde{M}_{0,\,n} \\
{\scriptstyle s_n}\downarrow & & \downarrow \\
\mathbb{P}^n \setminus R_n & \xrightarrow{\pi_n} & M_{0,\,n}
\end{array} \tag{3}
$$

The cross–ratios $\delta_i(z) = (z_1,\, z_2;\, z_3,\, z_i)$, where $z = (z_1, \ldots, z_n) \in (\mathbb{P}^1)^n$ and $4 \le i \le n$, define a morphism

$$\delta^{(n)} = (\delta_4,\, \ldots,\, \delta_n) \,:\, (\mathbb{P}^1)^n \setminus D_n \to (\mathbb{C}^{**})^{n-3} \setminus D_{n-3}$$

where $\mathbb{C}^{**} := \mathbb{P}^1 \setminus \{0,\, 1,\, \infty\}$. By the invariance of cross–ratio $\delta^{(n)}$ is constant along the orbits of the action of $\mathbb{P}GL(2, \mathbb{C})$ on $(\mathbb{P}^1)^n \setminus D_n$. Therefore, it factorizes through a mapping of the orbit space $\tilde{M}_{0,\,n} \to (\mathbb{C}^{**})^{n-3} \setminus D_{n-3}$. On the other hand, for each point $z \in (\mathbb{P}^1)^n \setminus D_n$ its $\mathbb{P}GL(2, \mathbb{C})$–orbit O_z contains the unique point z' of the form $z' = (0,\, 1,\, \infty,\, z'_4,\, \ldots,\, z'_n)$. This defines a regular section $\tilde{M}_{0,\,n} \to (\mathbb{P}^1)^n \setminus D_n$, and its image coincides with the image of the biregular embedding

$$(\mathbb{C}^{**})^{n-3} \setminus D_{n-3} \ni u = (u_4,\, \ldots,\, u_n) \longmapsto (0,\, 1,\, \infty,\, u_4,\, \ldots,\, u_n) \in (\mathbb{P}^1)^n \setminus D_n \;.$$

This shows that the above mapping $\tilde{M}_{0,\,n} \to (\mathbb{C}^{**})^{n-3} \setminus D_{n-3}$ is an isomorphism.

In the sequel we treat $\mathbb{P}^n$ as the projectivized space of the binary forms of degree n in u and v. For instance, $e_k = (0 : \ldots : 0 : 1_k : 0 : \ldots : 0) \in \mathbb{P}^n$ corresponds to the forms $cu^{n-k}v^k$, where $c \in \mathbb{C}^*$. Denote by O_q the $\mathbb{P}GL(2, \mathbb{C})$–orbit of a point $q \in \mathbb{P}^n$; it is a smooth quasiprojective variety. If the form q has the roots $z_1,\, z_2, \ldots$ of multiplicities $m_1,\, m_2,\, \ldots$, then we say that O_q is an orbit of type $O_{m_1,\, m_2, \ldots}$; furthermore, even in the case when O_q is not the only orbit of this type, without abuse of notation we often write $O_{m_1,\, m_2, \ldots}$ for the orbit O_q itself. Clearly, $O_{e_i} = O_{e_{n-i}}$, $i = 0, \ldots, n$; $O_{e_0} = O_n$ is the only one–dimensional orbit and, at the same time, the only closed orbit; $O_{e_i} = O_{n-i,\, i}$, $i = 1, \ldots, [n/2]$, are the only two-dimensional orbits, and any other orbit $O_q = O_{m_1,\, m_2,\, m_3, \ldots}$ has dimension 3. Its closure $\bar{O}_q$ is the union of the orbits O_q, O_n and $O_{m_i,\, n-m_i}$, $i = 1, 2, \ldots$ [AlFa, Proposition 2.1]. Furthermore, for any point $q \in \mathbb{P}^n \setminus R_n$, i.e. for any binary form q without multiple roots, its orbit $O_q = O_{1,\,1, \ldots, 1}$ is closed in $\mathbb{P}^n \setminus R_n$, and its closure in $\mathbb{P}^n$ is $\bar{O}_q = O_q \cup S_1$, where

$S_1 := O_n \cup O_{n-1,1} = \bar{O}_q \cap R_n$. Therefore, any Zariski closed subvariety Z of $I\!P^n$ such that $\dim (O_q \cap Z) > 0$ must meet the surface S_1. These observations yield the following lemma. [1]

2.1. Lemma. *If a linear subspace L in $I\!P^n$ does not meet the surface $S_1 = \bar{O}_{n-1,1} \subset R_n$, then it has at most finite intersection with any of the orbits O_q, where $q \in I\!P^n \backslash R_n$. In particular, this is so for a generic linear subspace L in $I\!P^n$ of codimension at least 3.*

c) Background in hyperbolic complex analysis

The next statement follows from Theorem 2.5 in [Za1].

2.2. Proposition. *Let $C \subset I\!P^2$ be a curve such that the Riemann surface $\operatorname{reg} C := C \setminus \operatorname{sing} C$ is hyperbolic and $I\!P^2 \setminus C$ is Brody hyperbolic, i.e. it does not contain any entire curve. Then $I\!P^2 \setminus C$ is Kobayashi complete hyperbolic and hyperbolically embedded into $I\!P^2$.*

We say that a complex space X is *almost* resp. *weakly Carathéodory hyperbolic* if for any point $p \in X$ there exist only finitely many resp. countably many points $q \in X$ which cannot be separated from p by bounded holomorphic functions. It will be called *almost* resp. *weakly C–hyperbolic* if X has a covering $Y \to X$, where Y is almost resp. weakly Carathéodory hyperbolic.

The next lemma easily follows from the definitions.

2.3. Lemma. *Let $f : Y \to X$ be a holomorphic mapping of complex spaces. If f is injective (resp. has finite resp. at most countable fibres) and X is C–hyperbolic (resp. almost resp. weakly C–hyperbolic), then Y is C–hyperbolic (resp. almost resp. weakly C–hyperbolic).*

[1] We are gratefull to H. Kraft who pointed out to us the approach used in the proof, and to M. Brion for mentioning to us the paper [AlFa].

3 Proof of Theorem 1.1 and a generalization

Proof of Theorem 1.1, a)

Let $\rho_C : \mathbb{P}^2 \to S^n(C^*_{norm})$ be the Zariski embedding introduced in section 2.a). The assumption that C^* is an immersed curve easily implies that $\rho_C^{-1}(R_n) = C$. The covering $s_n : (C^*_{norm})^n \setminus D_n \to S^n(C^*_{norm}) \setminus R_n$ is non–ramified. Thus, we arrive at the commutative diagram

$$\begin{array}{ccccc} Y & \overset{\tilde{\rho}_C}{\hookrightarrow} & (C^*_{norm})^n \setminus D_n & \hookrightarrow & (C^*_{norm})^n \\ \tilde{s}_n \downarrow & & s_n \downarrow & & \\ \mathbb{P}^2 \setminus C = X & \overset{\rho_C}{\hookrightarrow} & S^n(C^*_{norm}) \setminus R_n & & \end{array} \tag{4}$$

where $\tilde{s}_n : Y \to X$ is the induced covering. If the genus $g(C^*) \geq 2$, then $(C^*_{norm})^n$ has the polydisc Δ^n as the universal covering. Passing to the induced covering $Z \to Y$ we can extend (4) to the diagram

$$\begin{array}{ccc} Z & \hookrightarrow & \Delta^n \\ \downarrow & & \downarrow \\ Y & \overset{\tilde{\rho}_C}{\hookrightarrow} & (C^*_{norm})^n \\ \tilde{s}_n \downarrow & & \downarrow s_n \\ \mathbb{P}^2 \setminus C = X & \overset{\rho_C}{\hookrightarrow} & S^n(C^*_{norm}) \end{array} \tag{5}$$

Being a submanifold of the polydisc, Z is Carathéodory hyperbolic, and so X is C–hyperbolic. Therefore, we have proved the Theorem in the case $g \geq 2$.

Next we consider the case $g = 1$. Denote $E = C^*_{norm}$. Note that both $E^n \setminus D_n$ and $S^n E \setminus R_n$ are not C–hyperbolic or even hyperbolic, and so we can not apply the same arguments as above.

Represent E as $E = J(E) = \mathbb{C}/\Lambda_\omega$, where Λ_ω is the lattice generated by 1 and $\omega \in \mathbb{C}_+$ (here $\mathbb{C}_+ := \{z \in \mathbb{C} \,|\, \mathrm{Im} z > 0\}$). By Abel's Theorem (see e.g. [GH, 2.2]) we may assume this identification of E with its jacobian $J(E)$ being chosen in such a way that the image $\rho_C(\mathbb{P}^2)$ is contained in the hypersurface $s_n(H_0) = \phi_n^{-1}(\bar{0}) \cong$

$\mathbb{P}^{n-1} \subset S^n E$, where

$$H_0 := \{z = (z_1, \ldots, z_n) \in E^n \mid \sum_{i=1}^{n} z_i = 0\}$$

is an abelian subvariety in E^n and $\phi_n : S^n E \to J(E)$ denotes the n-th Abel–Jacobi map. The universal covering $\tilde{H}_0$ of H_0 can be identified with the hyperplane $\sum\limits_{i=1}^{n} x_i = 0$ in $\mathbb{C}^n = \tilde{E}^n$.

Consider the countable families $\tilde{D}_{ij}$ of parallel affine hyperplanes in $\mathbb{C}^n$ given by the equations $x_i - x_j \in \Lambda_\omega\,,\ \ i, j = 1, \ldots, n,\ i < j$.

Claim. The domain $\tilde{H}_0 \setminus \bigcup\limits_{i=1}^{n-1} \tilde{D}_{i,i+1}$ *is biholomorphic to* $(\mathbb{C} \setminus \Lambda_\omega)^{n-1}$.

Indeed, put $y_k := (x_k - x_{k+1}) \,|\, \tilde{H}_0\,,\ i = 1, \ldots, n-1$. It is easily seen that $(y_1, \ldots, y_{n-1}) : \tilde{H}_0 \to \mathbb{C}^{n-1}$ is a linear isomorphism whose restriction yields a biholomorphism as in the claim.

The universal covering of $(\mathbb{C} \setminus \Lambda_\omega)^{n-1}$ is the polydisc Δ^n, and so $(\mathbb{C} \setminus \Lambda_\omega)^{n-1}$ is C–hyperbolic. Put $\tilde{D}_n := \bigcup\limits_{i,j=1,\ldots,n} \tilde{D}_{ij}$. The open subset $\tilde{H}_0 \setminus \tilde{D}_n$ of $\tilde{H}_0 \setminus \bigcup\limits_{i=1}^{n-1} \tilde{D}_{i,i+1} \cong (\mathbb{C} \setminus \Lambda_\omega)^{n-1}$ is also C–hyperbolic (see (2.3)).

Denote by p the universal covering map $\mathbb{C}^n \to (\mathbb{C}/\Lambda_\omega)^n$. The restriction

$$p \,|\, \tilde{H}_0 \setminus \tilde{D}_n : \tilde{H}_0 \setminus \tilde{D}_n \to H_0 \setminus D_n \subset E^n \setminus D_n$$

is also a covering map. Therefore, $H_0 \setminus D_n$ is C–hyperbolic, and so $s_n(H_0) \setminus R_n$ is C–hyperbolic, too. Since $\rho_C \,|\, (\mathbb{P}^2 \setminus C) : \mathbb{P}^2 \setminus C \to s_n(H_0) \setminus R_n$ is an injective holomorphic mapping, by Lemma 2.3 $\mathbb{P}^2 \setminus C$ is C–hyperbolic. □

Proof of Theorem 1.1.b)

If $C \subset \mathbb{P}^2$ is a rational curve of degree $d > 1$, then $C^*_{norm} \cong \mathbb{P}^1$, $S^n \mathbb{P}^1 \cong \mathbb{P}^n$, and hence the Zariski embedding ρ_C embeds $\mathbb{P}^2$ into $\mathbb{P}^n \cong S^n \mathbb{P}^1$, where $n = \deg C^*$. The normalization map $\nu : \mathbb{P}^1 \to C^* \subset \mathbb{P}^2$ can be given as $\nu = (g_0 : g_1 : g_2)$, where $g_i(z_0, z_1) = \sum\limits_{j=0}^{n} b_j^{(i)} z_0^{n-j} z_1^j\,,\ i = 0, 1, 2$, are homogeneous polynomials of degree n without common factor.

If $x = (x_0 : x_1 : x_2) \in \mathbb{P}^2$ and $l_x \subset \mathbb{P}^{2*}$ is the dual line, then $\rho_C(x) = \nu^*(l_x) \in S^n\mathbb{P}^1 = \mathbb{P}^n$ is defined by the equation $\sum_{i=0}^{2} x_i g_i(z_0 : z_1) = 0$. Thus, $\rho_C(x) = (a_0(x) : \ldots : a_n(x))$, where $a_j(x) = \sum_{i=0}^{2} x_i b_j^{(i)}$.

Therefore, in the case of a plane rational curve C the Zariski embedding $\rho_C : \mathbb{P}^2 \to \mathbb{P}^n$ is a linear embedding given by the $3 \times (n+1)$–matrix $B_C := (b_j^{(i)})$, $i = 0, 1, 2$, $j = 0, \ldots, n$. In what follows we denote by $\mathbb{P}^2_C$ the image $\rho_C(\mathbb{P}^2)$, which is a plane in $\mathbb{P}^n$. This plane is generic iff the matrix B_C is generic, and this is the case iff C^* is a generic rational nodal curve. From Lemma 2.1 we get that for a generic rational nodal curve C^* the plane $\mathbb{P}^2_C$ has at most finite intersection with any of the $\mathbb{P}GL(2, \mathbb{C})$–orbits O_q, where $q \in \mathbb{P}^n \setminus R_n$.

Consider the following commutative diagram of morphisms

$$\begin{array}{ccccccc} Y & \xrightarrow{\tilde{\rho}_C} & (\mathbb{P}^1)^n \setminus D_n & \xrightarrow{\tilde{\pi}_n} & (\mathbb{C}^{**})^{n-3} \setminus D_{n-3} \hookrightarrow (\mathbb{C}^{**})^{n-3} \\ \tilde{s}_n \downarrow & & s_n \downarrow & & \downarrow \\ \mathbb{P}^2 \setminus C = X & \xrightarrow{\rho_C} & \mathbb{P}^n \setminus R_n & \xrightarrow{\pi_n} & M_{0,n} \end{array} \qquad (6)$$

where $\tilde{s}_n : Y \to X$ is the induced covering (cf. section 2.b), especially (3) above).

The mapping $\pi_n \circ \rho_C : X \to M_{0,n}$ has finite fibres. Hence, the same is valid for the mapping $\tilde{\pi}_n \circ \tilde{\rho}_C : Y \to (\mathbb{C}^{**})^{n-3} \setminus D_{n-3}$. By Lemma 2.3 Y, and thus also X, are almost C–hyperbolic. □

A generalization

If C^* has cusps, denote by L_C the union of their dual lines in $\mathbb{P}^2$. Clearly, L_C consists of the inflexional tangents of C and the cuspidal tangents at those cusps of C which are not simple, i.e. which can not be resolved by just one blow-up. Due to some analogy in tomography, we call L_C *the artifacts* of C. These artifacts arise naturally as soon as C^* is not immersed any more, namely we have:

$$\mathbb{P}^2_C \cap R_n = C \cup L_C \ .$$

In this more general situation we proved in [DZ] the following

3.2. Theorem. *Let* $C \subset \mathbb{P}^2$ *be an irreducible curve of genus* g. *Put* $n = \deg C^*$ *and* $X = \mathbb{P}^2 \setminus (C \cup L_C)$.
a) If $g \geq 1$, *then* X *is C–hyperbolic. If* $g = 0$, *then* X *is almost C–hyperbolic if at least one of the following conditions is fulfilled:*
i) $i(T_{p^*}A^*,\, A^*;\, p^*) \leq n-2$ *for any local analytic branch* $(A^*,\, p^*)$ *of* C^**;*
ii) C^* *has a cusp and it is not projectively equivalent to one of the curves* $(1 : g(t) : t^n)$, $(t : g(t) : t^n)$, *where* $g \in \mathbb{C}[t]$ *and* $\deg g \leq n-2$.
b) Let, furthermore, C^* *be an immersed curve. If* $g \geq 1$, *then* $\mathbb{P}^2 \setminus C$ *is C–hyperbolic. If* $g = 0$ *and i) is fulfilled, then* $\mathbb{P}^2 \setminus C$ *is almost C–hyperbolic; in particular, this is so if* C^* *is a generic rational nodal curve of degree* $n \geq 5$. *In both cases* $\mathbb{P}^2 \setminus C$ *is Kobayashi complete hyperbolic and hyperbolically embedded into* $\mathbb{P}^2$ (cf. (1.1).

Here $i(.,.;.)$ denotes the local intersection multiplicity.
In the case $g \geq 1$ the proof is the same as that of Theorem 1.1, and under the assumption i) the proof is an easy generalization. However, under the assumption ii) it is based on another idea.

Due to Lin's Theorem mentioned in the Introduction [Li, Thm. 13], we obtain the following

3.3 Corollary. *If* $C \subset \mathbb{P}^2$ *is one of the curves mentioned in Theorem 3.2, b) above, then the group* $\pi_1(\mathbb{P}^2 \setminus C)$ *is not almost nilpotent.*

Observe that an alternative way to prove this corollary might be by using an explicit presentation of the group $\pi_1(\mathbb{P}^2 \setminus C)$. Such a presentation, in the case of C^* being a generic nodal curve of genus 0 or 1, was found by Zariski [Zar], and for generic nodal curves of any genus by J. Kaneko [Ka]. From these presentations it can be checked rather easily that these groups are not almost nilpotent.

Furthermore, if a curve is a specialization of a curve with nodal dual, then the fundamental group of the complement has an epimorphism onto one of the groups in the Zariski–Kaneko list, and therefore it is not almost nilpotent, too. Using the same type of arguments as in [No, 3.5] one can prove that any irreducible immersed plane curve is a specialization of an irreducible nodal curve with the same degree and geometric genus. It remains to pass to the dual curves, which are also of equal degrees, due to the class formula.

The next Proposition shows that the examples in the next section, obtained by applying Theorems 1.1 and 3.2, are, indeed, at the borderline, as far as the C-hyperbolicity is concerned.

3.4. Proposition. *Let $C \subset \mathbb{P}^2$ be an irreducible curve which has C-hyperbolic complement. Then we have:*
a) C is of degree at least 6;
b) C has worse singularities than ordinary double points.

To prove part a) we first mention that for curves of degree ≤ 4 the complement is not even hyperbolic, since there always exist projective lines which intersect C at most in two points, see [Gr2]. The same remains true for the irreducible quintics which are not Plücker, see [DZ, Lemma 4.9]. For irreducible Plücker quintics the fundamental group of the complement is abelian [Deg 1,2], and so it is isomorphic to $\mathbb{Z}/5\mathbb{Z}$. Thus, the only coverings Y over $\mathbb{P}^2 \setminus C$ are finite cyclic ones. Being quasiprojective, such an Y is a Liouville variety, and hence $\mathbb{P}^2 \setminus C$ is not C–hyperbolic.

The proof of b) is done by the same arguments. Indeed, by the Deligne-Fulton Theorem on abelianess of the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ for a nodal curve C, this group is finite cyclic if C is also irreducible. Hence, any covering over $\mathbb{P}^2 \setminus C$ is a Liouville one. $\square$

4 Examples

In this section we give explicit examples of plane curves with C–hyperbolic complements. Furthermore, we construct, for every even $d \geq 6$, families of irreducible curves in $\mathcal{H}(d)$, especially of elliptic or rational such curves. They are described by the degree, the genus and the singularities of their members or of the dual curves. Most of them arise from Theorem 1.1 in the special case where C^* is a nodal curve, and only the case of maximal cuspidal rational sextics has to be treated in a different way (see

Proposition 4.5 below).

Examples of curves of genus $g \geq 2$

Let $C \subset \mathbb{P}^2$ be an irreducible curve of genus $g \geq 2$ whose dual C^* is a nodal curve of degree $n \geq 4$ with δ nodes. Such a curve C does exist for any given δ with $0 \leq \delta \leq \frac{(n-1)(n-2)}{2} - 2$ (see [Se, §11, p.347]; [O, (6.7)]). By the class formula (1) and the genus formula C has degree $d = n(n-1) - 2\delta$, which can be any even integer from the interval $[2(n+1), n(n-1)]$. Applying Theorem 1.1 we obtain the following

4.1. Proposition. *For any given $d \geq 10$ there exist $g \geq 2$, $n \geq 4$ and $\delta \geq 0$ such that the family of nodal curves C^* of genus g, degree n and with δ nodes is non-empty and their dual curves C have degree d and belong to $\mathcal{H}(d)$. Moreover, the complements $\mathbb{P}^2 \setminus C$ are C-hyperbolic.*

Observe that, as it was shown in [Gr1, CaGr, GP], an irreducible plane curve C belongs to $\mathcal{H}(d)$ if the following three conditions hold:
(a) the geometric genus g of C is at least two,
(b) each tangent line to C^* intersects with C^* in at least two points, and
(c) $2n < d$, where as before $d = \deg C$ and $n = \deg C^*$.

These conditions are less restrictive than those of the above Proposition, since here C^* may possess some cusps. For such a C^* by the genus formula $2g \leq (n-1)(n-2)$, hence for $g \geq 2$ we have $n \geq 4$. Now, from c) we have $d \geq 9$. By the class formula this lower bound is really achieved for the family of duals of the irreducible plane quartics C^* with an ordinary cusp as the only singular point (see e.g. [Na, p.130]). However, we do not know whether in this example the complement of C is C-hyperbolic.

Examples of elliptic curves.

If the dual C^* of C is an immersed elliptic curve, then by the class formula (1) $d = \deg C = 2n \geq 6$, where $n = \deg C^* \geq 3$. Let C be a sextic in $\mathbb{P}^2$ with nine cusps. Then C is an elliptic Plücker curve whose dual C^* is a smooth cubic; vice versa, the dual curve of a smooth cubic is a sextic with nine ordinary cusps. From Theorem 1.1

we get the following

4.2. Proposition. *Every irreducible plane sextic with nine ordinary cusps is an elliptic curve with C-hyperbolic complement, which belongs to* $\mathcal{H}(6)$.

Note that up to projective equivalence the above family is one dimensional, see e.g. [BK]. We refere to [GKZ, I.2.E] for explicit Schläfli's equations of these sextics. For instance, the dual of the Fermat cubic $-x_0^3 + x_1^3 + x_2^3 = 0$ is the sextic

$$x_0^6 + x_1^6 + x_2^6 - 2x_0^3 x_1^3 - 2x_1^3 x_2^3 - 2x_0^3 x_2^3 = 0 \ .$$

Another example is the family of the dual curves C of the nodal quartics C^* with two nodes (for the existence see e.g. [Na, p.133]). Here $d = \deg C = 8$. Together with (4.1) this yields the following

4.3. Proposition. *For any even* $d \geq 6$ *there exist* $g \geq 1$, $n \geq 3$ *and* $\delta \geq 0$ *such that the family of the nodal curves* C^* *of genus* g, *degree* n *and with* δ *nodes is non-empty and the dual curves* C *have degree* d *and belong to* $\mathcal{H}(d)$. *Moreover, their complements* $\mathbb{P}^2 \setminus C$ *are C-hyperbolic.*

Examples of rational curves

The generic rational curve C^* of degree $n \geq 3$ is a nodal Plücker curve (see e.g. [Au]). Its dual curve C has even degree $d = 2(n-1)$ and $3(d-2)/2$ cusps. Vice versa, any rational Plücker curve C of even degree $d = 2(n-1)$ and with $3(n-2)$ cusps is dual to a nodal curve C^* of degree n. This is the maximal number of cusps which a rational curve of degree d can possess, and so these curves are called *maximal cuspidal rational curves* [Zar, p. 267]. Applying Theorem 1.1 we obtain the following

4.4. Proposition. *For any even degree* $d \geq 8$ *the generic rational maximal cuspidal plane curve of degree* d *belongs to* $\mathcal{H}(d)$ *and its complement is almost C-hyperbolic.* [2]

[2] Moreover, due to Theorem 3.2 a) i) we can drop here the assumption of genericity.

What happens with rational maximal cuspidal curves of lower degrees? For $d = 2$ resp. $d = 4$ we have a smooth conic resp. a three cuspidal quartic (which are both projectively unique [Na, p.146]). It is easily seen that their complements are not even Brody hyperbolic. It remains the case $d = 6$. In this case we have the following

4.5. Proposition. *The generic rational maximal cuspidal plane sextic C belongs to $\mathcal{H}(6)$.*

Proof. We keep the notation of section 2.b. From the proof of Theorem 1.1.b we know that C is a generic plane section of the discriminant hypersurface $R_4 \subset \mathbb{P}^4$. We have denoted this plane as $\mathbb{P}^2_C$. Clearly, being generic, $\mathbb{P}^2_C$ does not meet the one–dimensional orbit O_4. From the definition of the Zariski embedding it easily follows that it intersects the orbit closure $S_1 = \bar{O}_{3,1}$ resp. $S_2 := \bar{O}_{2,2}$ in the set $K = \{\text{the cusps of } C\}$ resp. $N = \{\text{the nodes of } C\}$. Therefore, it intersects the only 3-dimensional orbit $O_{2,1,1}$ contained in R_4 in the curve $C \setminus (K \cup N)$. By the Plücker formulas, $\operatorname{card} K = 6$ and $\operatorname{card} N = 4$ (this agrees with the fact that $\deg S_1 = 6$ and $\deg S_2 = 4$ [AlFa, Proposition 1.1]). Let $C_q = \mathbb{P}^2_C \cap \bar{O}_q$, where $O_q = O_{1,1,1,1}$, i.e. $q \in \mathbb{P}^4 \setminus R_4$. Since $\bar{O}_q = O_q \cup S_1$ (see section 2.b), it is easily seen that the curve $C_q \subset \mathbb{P}^2_C$ meets C exactly in the cusps of C.

Now we use the diagram (6). Let $f : \mathbb{C} \to \mathbb{P}^2_C \setminus C = \mathbb{P}^2_C \setminus R_4$ be an entire curve. Since $s_4 : (\mathbb{P}^1)^4 \setminus D_4 \to \mathbb{P}^4 \setminus R_4$ is an unramified covering, f can be lifted to $(\mathbb{P}^1)^4 \setminus D_4$. The curve $\mathbb{C}^{**}$ being hyperbolic, this lifted entire curve has to be contained in a fiber of $\tilde{\pi}_4$, which is an orbit of the $\mathbb{P}GL(2,\mathbb{C})$-action on $(\mathbb{P}^1)^n$. The map s_4 is equivariant with respect to $\mathbb{P}GL(2,\mathbb{C})$-actions on $(\mathbb{P}^1)^4 \setminus D_4$ and $\mathbb{P}^4 \setminus R_4$, respectively, whence the entire curve $f : \mathbb{C} \to \mathbb{P}^2_C \setminus C$ lies in a $\mathbb{P}GL(2,\mathbb{C})$–orbit, too.

Thus, to show that $\mathbb{P}^2 \setminus C$ is Brody hyperbolic it is enough to show that the non–compact curves $C_q \setminus C = O_q \cap \mathbb{P}^2_C$ are hyperbolic for all $q \in \mathbb{P}^4 \setminus R_4$. Once this is done, Proposition 4.5 follows from Proposition 2.2.

It is well known (see [H, p.58] or [PV] [3]) that the 3-dimensional $\mathbb{P}GL(2,\mathbb{C})$–orbit closures in $\mathbb{P}^4$ form a linear pencil. This pencil of sextic threefolds is generated by its members $3P$ and $2H$, where the irreducible quadric resp. cubic P and H are defined by the basic invariants $\tau_2 = a_0a_4 - 4a_1a_3 + 3a_2^2$ resp. $\tau_3 = a_0a_2a_4 - a_0a_3^2 - a_1^2a_4 +$

[3]We are grateful to R.O.Buchweitz for the last reference

$2a_1a_2a_3 - a_2^3$ (here we use the coordinates where $q(u,\, v) = a_0u^4 + 4a_1u^3v + 6a_2u^2v^2 + 4a_3uv^3 + a_4v^4$). The base point set of this linear pencil is the surface $S_1 = H \cap P$, as it follows from the description of the orbit closures in [AlFa] (see section 2.b).

The restriction of the above pencil to the plane $\mathbb{P}^2_C$ is the linear pencil of plane sextics $\alpha := (C_q)$ generated by $3p$ and $2h$, where $p := P \cap \mathbb{P}^2_C$ and $h := H \cap \mathbb{P}^2_C$ are respectively irreducible conic and cubic. Its base point set $p \cap h$ is the set K of cusps of C (note that C itself is a member of α). The intersection of p and h at the points of K is transversal, because $\operatorname{card} K = p \cdot h = 6$. Since the ideal generated by two distinct members $C' = C_{q'}$ and $C'' = C_{q''}$ is the same as the one generated by $3p$ and $2h$, we have for the local intersection multiplicities at any point $x \in K$

$$i(C',\, C'';\, x) = i(3p,\, 2h;\, x) = 6\, i(p,\, h;\, x) = 6\,.$$

Assume now that a member C_q of the pencil α has an irreducible component T which intersects C in at most two points $x',\, x'' \in K$. Since $i(T,\, C;\, x) \le 6$ for $x = x',\, x''$, we would have $\deg C {\cdot} \deg T \le 12$, and hence $\deg T \le 2$. If T would be a projective line, then by Bezout's Theorem $T \cdot C = \deg C = 6$, whence $i(T,\, C;\, x) = 3$ for $x = x',\, x''$, and so T should be a common cuspidal tangent of C at these two cusps $x',\, x''$, which is impossible for a Plücker curve C. If, further, T would be a smooth conic, then by Bezout's Theorem we would have $i(T,\, C;\, x) = 6$ for $x = x',\, x''$, again in contradiction with the fact that C is a Plücker curve. Indeed, an ordinary cusp $(C,\, P)$ can be uniformized by $t \longmapsto (t^2, t^3 + O(t^4))$, see e.g. [Na, 1.5.8], and therefore the local intersection multiplicity of an ordinary cusp with a smooth curve germ $(C',\, P)$ can be at most 3. To see this, observe that plugging this parametrization into the power series expansion at P of the defining equation of C', its linear term will contain a non–zero monomial in t of order at most 3, which cannot be cancelled by further higher order terms [4]. Thus, there is no irreducible component T as above, and hence all the non–compact curves $C_q \setminus C$ in $\mathbb{P}^2_C \setminus C$ are hyperbolic. □

4.6. Remark. The dimension of the family of the plane rational nodal curves of degree $n \ge 3$ modulo the projective equivalence is $3(n-3) = \frac{3}{2}(d-4)$, where $d = 2(n-1)$ is the degree of the dual curves. In particular, the curves from the above Proposition form a 3–dimensional family.

[4] see also [FZ, (1.4)] for a more general fact.

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