Holomorphic curves into algebraic varieties intersecting moving hypersurface targets

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Abstract

In [Ann. of Math.169 (2009)], Min Ru proved a second main theorem for algebraically nondegenerate holomorphic curves in complex projective varieties intersecting fixed hypersurface targets. In this paper, by introducing a new proof method for the case of projective varieties, we generalize this result to moving hypersurface targets.

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1 Introduction

During the last century, several Second Main Theorems have been established for linearly nondegenerate holomorphic curves in complex projective spaces intersecting (fixed or moving) hyperplanes, and we now have a satisfactory knowledge about it. Motivated by a paper of Corvaja-Zannier [6] in Diophantine approximation, in 2004 Ru [20] proved a Second Main Theorem for algebraically nondegenerate holomorphic curves in the complex projective space $\mathbb{CP}^n$ intersecting (fixed) hypersurface targets, which settled a long-standing conjecture of Shiffman [22]. In 2011, Dethloff-Tan [7] generalized this result of Ru to moving hypersurface targets (this means where the coefficients of the hypersurfaces are meromorphic functions) in $\mathbb{CP}^n$. The counterpart of the Second Main Theorem of Dethloff-Tan in Diophantine approximation was independently given by Chen-Ru-Yan [5] and Le [14] in 2015.

In order to reduce the case of hypersurfaces to the case of hyperplanes by an approximation, the above authors construct a filtered vector space. The key point in their proof is the following property: If homogeneous polynomials $Q_0, \dots, Q_n$ in $\mathbb{C}[x_0, \dots, x_n]$ have no non-trivial common solutions, then $\{Q_0, \dots, Q_n\}$ is a regular sequence. Thanks to this property, they can construct linear isomorphisms showing that the dimension of all factor vector spaces in the filtration is exactly equal to the corresponding value of the Hilbert polynomial of a common algebraic variety, and then can be calculated. However, this regular sequence property is not true any more for the general case of varieties $V \subset \mathbb{CP}^M$, and is related to whether or not the homogenous coordinate ring of V is Cohen-Macauley, moreover, there are examples to show that their linear isomorphisms can not be extended to the general case of varieties. So by dropping this restriction on the variety V and thereby losing regular sequences, we can no longer exactly calculate the dimensions of the various factor vector spaces of the filtration by using this method.

The result of Corvaja-Zennier [6] was actually reproved and generalized to the case of projective varieties by Evertse-Ferretti [11] using somewhat different ideas. Their approach also admitted extensions in Nevanlinna theory which were developped again by Ru [21] in 2009, obtaining a Second Main Theorem for entire curves in arbitrary projective varieties V, with respect to hypersurfaces of the ambient space. To prove the Second Main Theorem, in [21], Ru uses the finite morphism $\phi : V \to \mathbb{CP}^{q-1}$, $\phi(x) := [Q_1(x) : \cdots : Q_q(x)]$, where the Q_j's are homogeneous polynomials (with common degree) defining the given hypersurfaces. Thanks to this finite morphism, he can use a generalization of Mumford's identity (the version with explicit estimates obtained by Evertse and Ferretti [10, 11]) for the variety $\mathrm{Im}\phi \subset \mathbb{CP}^{q-1}$. However, for the case of moving hypersurfaces, we do not have such a morphism.

The purpose of this article is twofold, giving a Second Main Theorem for entire curves in projective varieties intersecting moving hypersurfaces and introducing a new approach for the case of projective varieties. Firstly, we show that by specializing the coefficients of the polynomials corresponding to the moving hypersufaces in generic points, the dimensions of the given vector spaces do not change. Secondly, we construct a filtration of the vector space corresponding to the coordinate ring of the variety. After that, by observing the Hilbert sequence asymptotics, we calculate the sum of the dimensions of all the factors of the vector spaces in the filtration and by using the algebraic properties of our filtration, properties of its Hilbert function and also techniques in combinatorics, we prove that almost all of these factor vector spaces have the same dimension. Finally, we prove that we can neglect the other factors vector spaces of the filtration where the dimension is not as expected. Another difficulty in the case of moving hypersurface targets is that they are in general position only for generic points. In order to overcome this difficulty, we use an element of the ideal of the inertia forms of the system of polynomials defining the moving hypersurfaces in order to control the locus where the divisors are not in general position. In fact, the ideal of the inertia forms of such a system of polynomials is not a principal ideal in general (unless V is a complete intersection variety). But we will show that there exists an element of this ideal with properties which are enough for our purpose.

Our method was used again by Ji-Yan-Yu [13], Yan-Yu [26], Son-Tan-Thin [23] to prove their Second Main Theorems and Schmidt's Subspace Theorems.

Let f be a holomorphic mapping of $\mathbb{C}$ into $\mathbb{CP}^M$, with a reduced representation $f := (f_0 : \cdots : f_M)$. The characteristic function $T_f(r)$ of f is defined by

$$T_f(r) := \frac{1}{2\pi}\int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta, \quad r > 1,$$

where $\|f\| := \max\{|f_0|, \ldots, |f_M|\}$.

Let ν be a divisor on $\mathbb{C}$. The counting function of ν is defined by

$$N_\nu(r) := \int_1^r \log \frac{\sum_{|z|<t} \nu(z)}{t} dt, \quad r > 1.$$

For a non-zero meromorphic function φ, denote by ν_φ the zero divisor of φ, and set $N_\varphi(r) := N_{\nu_\varphi}(r)$. Let Q be a homogeneous polynomial in the variables $x_0, \ldots, x_M$ with coefficients which are meromorphic functions. If $Q(f) := Q(f_0, \ldots, f_M) \not\equiv 0$, we define $N_f(r, Q) := N_{Q(f)}(r)$. Denote by $Q(z)$ the homogeneous polynomial over $\mathbb{C}$ obtained by evaluating the coefficients of Q at a specific point $z \in \mathbb{C}$ in which all coefficient functions of Q are holomorphic (in particular $Q(z)$ can be the zero polynomial).

We say that a meromorphic function φ on $\mathbb{C}$ is "small" with respect to f if $T_\varphi(r) = o(T_f(r))$ as $r \to \infty$ (outside a set of finite Lebesgue measure).

Denote by $\mathcal{K}_f$ the set of all "small" (with respect to f) meromorphic functions on $\mathbb{C}$. Then $\mathcal{K}_f$ is a field.

For a positive integer d, we set

$$\mathcal{T}_d := \big\{(i_0, \dots, i_M) \in \mathbb{N}_0^{M+1} : i_0 + \cdots + i_M = d\big\}.$$

Let $\mathcal{Q} = \{Q_1, \dots, Q_q\}$ be a set of $q \geq n+1$ homogeneous polynomials in $\mathcal{K}_f[x_0, \dots, x_M]$, $\deg Q_j = d_j \geq 1$. We write

$$Q_j = \sum_{I \in \mathcal{T}_{d_j}} a_{jI} x^I \quad (j = 1, \dots, q)$$

where $x^I = x_0^{i_0} \cdots x_M^{i_M}$ for $x = (x_0, \dots, x_M)$ and $I = (i_0, \dots, i_M)$. Denote by $\mathcal{K}_{\mathcal{Q}}$ the field over $\mathbb{C}$ of all meromorphic functions on $\mathbb{C}$ generated by $\big\{a_{jI} : I \in \mathcal{T}_{d_j}, j \in \{1, \dots, q\}\big\}$. It is clearly a subfield of $\mathcal{K}_f$.

Let $V \subset \mathbb{CP}^M$ be an arbitrary projective variety of dimension n, generated by the homogeneous polynomials in its ideal $\mathcal{I}(V)$. Assume that f is nonconstant and $\mathrm{Im} f \subset V$. Denote by $\mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)$ the ideal in $\mathcal{K}_{\mathcal{Q}}[x_0, \dots, x_M]$ generated by $\mathcal{I}(V)$. Equivalently $\mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)$ is the (infinite-dimensional) $\mathcal{K}_{\mathcal{Q}}$-subvector space of $\mathcal{K}_{\mathcal{Q}}[x_0, \dots, x_M]$ generated by $\mathcal{I}(V)$. We note that $Q(f) \equiv 0$ for every homogeneous polynomial $Q \in \mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)$. We say that f is algebraically nondegenerate over $\mathcal{K}_{\mathcal{Q}}$ if there is no homogeneous polynomial $Q \in \mathcal{K}_{\mathcal{Q}}[x_0, \dots, x_M] \setminus \mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)$ such that $Q(f) \equiv 0$.

The set $\mathcal{Q}$ is said to be $V-$ admissible (or in (weakly) general position (with respect to V)) if there exists $z \in \mathbb{C}$ in which all coefficient functions of all Q_j, $j = 1, ..., q$ are holomorphic and such that for any $1 \leqslant j_0 < \cdots < j_n \leqslant q$ the system of equations

$$\begin{cases} Q_{j_i}(z)(x_0, \dots, x_M) = 0 \\ \qquad 0 \leqslant i \leqslant n \end{cases} \tag{1.1}$$

has no solution $(x_0, \dots, x_M)$ satisfying $(x_0 : \cdots : x_M) \in V$. As we will show in section 2, in this case this is true for all $z \in \mathbb{C}$ excluding a discrete subset of $\mathbb{C}$.

As usual, by the notation "$\| P$" we mean that the assertion P holds for all $r \in [1, +\infty)$ excluding a Borel subset E of $(1, +\infty)$ with $\int\limits_E dr < +\infty$.

Our main result is stated as follows:

Main Theorem. *Let $V \subset \mathbb{CP}^M$ be an irreducible (possibly singular) variety of dimension n, and let f be a non-constant holomorphic map of $\mathbb{C}$ into V. Let $\mathcal{Q} = \{Q_1, \dots, Q_q\}$ be a $V-$ admissible set of homogeneous polynomials in $\mathcal{K}_f[x_0, \dots, x_M]$ with $\deg Q_j = d_j \geq 1$. Assume that f is algebraically nondegenerate over $\mathcal{K}_{\mathcal{Q}}$. Then for any $\varepsilon > 0$,*

$$\|(q - n - 1 - \varepsilon)T_f(r) \leqslant \sum_{j=1}^{q} \frac{1}{d_j} N_f(r, Q_j).$$

In the special case where the coefficients of the polynomials Q_j's are constant and the variety V is smooth, the above theorem is the Second Main Theorem of Ru in [21]. According to Vojta ([25], p. 183) generalizing this theorem of Ru to singular varieties can be done already by his proof methods without essential changes of the proof (see also Chen-Ru-Yan [3], [4]), so the essential generalization in our main result is the one to moving targets.

We define the defect of f with respect to a homogenous polynomial $Q \in \mathcal{K}_f[x_0, \dots, x_M]$ of degree d with $Q(f) \not\equiv 0$ by

$$\delta_f(Q) := \lim_{r \to +\infty} \inf \Big(1 - \frac{N_f(r, Q)}{d \cdot T_f(r)}\Big).$$

As a corollary of the Main Theorem we get the following defect relation.

Corollary 1.1. *Under the assumptions of the Main theorem, we have*

$$\sum_{j=1}^{q} \delta_f(Q_j) \leqslant n + 1.$$

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2 Lemmas

Let $\mathcal{K}$ be an arbitrary field over $\mathbb{C}$ generated by a set of meromorphic functions on $\mathbb{C}$. Let V be a sub-variety in $\mathbb{CP}^M$ of dimension n defined by the homogeneous ideal $\mathcal{I}(V) \subset \mathbb{C}[x_0, \dots, x_M]$. Denote by $\mathcal{I}_{\mathcal{K}}(V)$ the ideal in $\mathcal{K}[x_0, \dots, x_M]$ generated by $\mathcal{I}(V)$.

For each positive integer k and for any (finite or infinite dimensional) $\mathbb{C}$-vector sub-space W in $\mathbb{C}[x_0, \dots, x_M]$ or for any $\mathcal{K}$-vector sub-space W in $\mathcal{K}[x_0, \dots, x_M]$, we denote by W_k the vector sub-space consisting of all homogeneous polynomials in W of degree k (and of the zero polynomial; we remark that W_k is necessarily of finite dimension).
The Hilbert polynomial H_V of V is defined by

$$H_V(N) := \dim_{\mathbb{C}} \frac{\mathbb{C}[x_0, \dots, x_M]_N}{\mathcal{I}(V)_N}, \quad N \in \mathbb{N}_0.$$

By the usual theory of Hilbert polynomials (see e.g. [12]), for $N >> 0$, we have

$$H_V(N) = \deg V \cdot \frac{N^n}{n!} + O(N^{n-1}).$$

Definition 2.1. *Let W be a $\mathcal{K}$-vector sub-space in $\mathcal{K}[x_0, \dots, x_M]$. For each $z \in \mathbb{C}$, we denote*

$$W(z) := \{P(z) : P \in W, \text{ all coefficients of } P \text{ are holomorphic at } z\}.$$

It is clear that $W(z)$ is a $\mathbb{C}$-vector sub-space of $\mathbb{C}[x_0, \dots, x_M]$.

Lemma 2.2. *Let W be a $\mathcal{K}$-vector sub-space in $\mathcal{K}[x_0, \dots, x_M]_N$. Assume that $\{h_j\}_{j=1}^K$ is a basis of W. Then $\{h_j(a)\}_{j=1}^K$ is a basis of $W(a)$ (and in particular $\dim_{\mathcal{K}} W = \dim_{\mathbb{C}} W(a)$) for all $a \in \mathbb{C}$ excluding a discrete subset.*

Proof. Let (c_{ij}) be the matrix of coefficients of $\{h_j\}_{j=1}^K$. Since $\{h_j\}_{j=1}^K$ are linearly independent over $\mathcal{K}$, there exists a square submatrix A of (c_{ij}) of order K and such that $\det A \not\equiv 0$. Let a be an arbitrary point in $\mathbb{C}$ such that $\det A(a) \neq 0$ and such that all coefficients of $\{h_j\}_{j=1}^K$ are holomorphic at a. For each $P \in W$ whose coefficients are all holomorphic at a, we write $P = \sum_{j=1}^K t_j h_j$ with $t_j \in \mathcal{K}$. In fact, there are coefficients b_j ($j = 1, \dots, K$) of

P such that $(t_1, \dots, t_K)$ is the unique solution in $\mathcal{K}^K$ of the following system of linear equations:

$$A \cdot \begin{pmatrix} t_1 \\ \cdot \\ \cdot \\ \cdot \\ t_K \end{pmatrix} = \begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_K \end{pmatrix}.$$

By our choice of a, so in particular we have $\det A(a) \neq 0$, and since $\{b_j\}_{j=1}^K$ are holomorphic at a, we get that the $\{t_j\}_{j=1}^K$ are holomorphic at a. Therefore, $P(a) = \sum_{j=1}^K t_j(a)h_j(a)$, $t_j(a) \in \mathbb{C}$. On the other hand, still by our choice of a, we have $h_j(a) \in W(a)$ for all $j \in \{1, \dots, K\}$. Hence, $\{h_j(a)\}_{j=1}^K$ is a generating system of $W(a)$. Since $\det A(a) \neq 0$, the matrix $(c_{ij}(a))$ has maximum rank. Therefore, $\{h_j(a)\}_{j=1}^K$ are also linearly independent over $\mathbb{C}$. $\square$

Throughout of this section, we consider a $V-$ admissible set of $(n+1)$ homogeneous polynomials $Q_0, \dots, Q_n$ in $\mathcal{K}[x_0, \dots, x_M]$ of common degree d. We write

$$Q_j = \sum_{I \in \mathcal{T}_d} a_{jI} x^I, \quad (j = 0, \dots, n),$$

where $a_{jI} \in \mathcal{K}$ and $\mathcal{T}_d$ is again the set of all $I := (i_0, \dots, i_M) \in \mathbb{N}_0^{M+1}$ with $i_0 + \dots + i_M = d$.
Let $t = (\dots, t_{jI}, \dots)$ be a family of variables. Set

$$\widetilde{Q_j} = \sum_{I \in \mathcal{T}_d} t_{jI} x^I \in \mathbb{C}[t, x], \quad (j = 0, \dots, n).$$

We have

$$\widetilde{Q_j}(\dots, a_{jI}(z), \dots, x_0, \dots, x_M) = Q_j(z)(x_0, \dots, x_M).$$

Assume that the ideal $\mathcal{I}(V)$ is generated by homogeneous polynomials $P_1, \dots, P_m$. Since $\{Q_0, \dots, Q_n\}$ is a $V-$ admissible set, there exists $z_0 \in \mathbb{C}$ such that the homogeneous polynomials $P_1, \dots, P_m, Q_0(z_0), \dots, Q_n(z_0)$ in $\mathbb{C}[x_0, \dots, x_M]$ have no common non-trivial solutions. Denote by ${}_{\mathbb{C}[t]}(P_1, \dots, P_m, \widetilde{Q_0}, \dots, \widetilde{Q_n})$ the ideal in the ring of polynomials in $x_0, \dots, x_M$ with coefficients in $\mathbb{C}[t]$ generated by $P_1, \dots, P_m, \widetilde{Q_0}, \dots, \widetilde{Q_n}$. A polynomial $\widetilde{R}$ in $\mathbb{C}[t]$ is called an *inertia form* of the polynomials $P_1, \dots, P_m, \widetilde{Q_0}, \dots, \widetilde{Q_n}$ if it has the following property (see e.g. [27]):

$$x_i^s \cdot \widetilde{R} \in \ _{\mathbb{C}[t]}(P_1, \dots, P_m, \widetilde{Q_0}, \dots, \widetilde{Q_n})$$

for $i = 0, \dots, M$ and for some non-negative integer s.

It is well known that for the $(m + n + 1)$ homogeneous polynomials $P_i(x_0, \dots, x_M)$, $\widetilde{Q_j}(\dots, t_{jI}, \dots, x_0, \dots, x_M)$, $i \in \{1, \dots, m\}$, $j \in \{0, \dots, n\}$ there exist finitely many inertia forms $\widetilde{R}_1, ..., \widetilde{R}_s$ (which are homogeneous polynomials in the t_{jI} separately for each j $(j = 0, \dots, n)$) such that the following holds : For special values t^0_{jI} of t_{jI} the $(m + n + 1)$ homogeneous polynomials $P_i(x_0, \dots, x_M)$, $\widetilde{Q_j}(\dots, t^0_{jI}, \dots, x_0, \dots, x_M)$, $i \in \{1, \dots, m\}$, $j \in \{0, \dots, n\}$ have a common non-trivial solution in $x_0, \dots, x_M$ if and only if t^0_{jI} is a common zero of the inertia forms $\widetilde{R}_1, ..., \widetilde{R}_s$ (see e.g. [12], page 35 or [27], page 254). Choose such a $\widetilde{R}$ for the special values $t^0_{jI} = a_{jI}(z_0)$, and put $R(z) := \widetilde{R}(\dots, a_{kI}(z), \dots) \in \mathcal{K}$. Then by construction, $R(z_0) \neq 0$, hence $R \in \mathcal{K} \setminus \{0\}$, so in particular R only vanishes on a discrete subset of $\mathbb{C}$, and, by the above property of the inertia form $\widetilde{R}$, outside this discrete subset, $Q_0(z), \dots, Q_n(z)$ have no common solutions in V. Furthermore, by the definition of the inertia forms, there exists a non-negative integer s such that

$$x_i^s \cdot R \in \ _{\mathcal{K}}(P_1, \dots, P_m, Q_0, \dots, Q_n), \text{ for } i = 0, \dots, M, \tag{2.1}$$

where $_{\mathcal{K}}(P_1, \dots, P_m, Q_0, \dots, Q_n)$ is the ideal in $\mathcal{K}[x_0, \dots, x_M]$ generated by $P_1, \dots, P_m, Q_0, \dots, Q_n$.

Let f be a nonconstant meromorphic map of $\mathbb{C}$ into $\mathbb{CP}^M$. Denote by $\mathcal{C}_f$ the set of all non-negative functions $h : \mathbb{C} \longrightarrow [0, +\infty] \subset \overline{\mathbb{R}}$, which are of the form

$$\frac{|u_1| + \cdots + |u_k|}{|v_1| + \cdots + |v_\ell|}\,, \tag{2.2}$$

where $k, \ell \in \mathbb{N}$, $u_i, v_j \in \mathcal{K}_f \setminus \{0\}$.

By the First Main Theorem we have

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+|\phi(re^{i\theta})|d\theta = o(T_f(r)), \quad \text{as } r \to \infty$$

for $\phi \in \mathcal{K}_f$. Hence, for any $h \in \mathcal{C}_f$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ h(re^{i\theta})d\theta = o(T_f(r)), \quad \text{as } r \to \infty.$$

It is easy to see that sums, products and quotients of functions in $\mathcal{C}_f$ are again in $\mathcal{C}_f$.

By the result on the inertia forms mentioned above, similarly to Lemma 2.2 in [7], we have

Lemma 2.3. *Let $\{Q_j\}_{j=0}^n$ be a $V-$ admissible set of homogeneous polynomials of degree d in $\mathcal{K}[x_0, \ldots, x_M]$. If $\mathcal{K} \subset \mathcal{K}_f$, then there exist functions $h_1, h_2 \in \mathcal{C}_f \setminus \{0\}$ such that,*

$$h_2 \cdot \|f\|^d \leqslant \max_{j\in\{0,\ldots,n\}} |Q_j(f_0, \ldots, f_M)| \leqslant h_1 \cdot \|f\|^d.$$

In fact, the second inequality is elementary. In order to obtain the first inequality, we use equation (2.1) in the same way as the corresponding equation in Lemma 2.1 in [7], and we observe that we have $P_i(f_0, ..., f_M) \equiv 0$ for $i = 1, ..., m$ since $f(\mathbb{C}) \subset V$, so the maximum only needs to be taken over the $Q_j(f_0, ..., f_M)$, $j = 0, ..., n$. The rest of the proof is identically to the one of Lemma 2.2 in [7].

We use the lexicographic order in $\mathbb{N}_0^n$ and for $I = (i_1, \ldots, i_n)$, set $\|I\| := i_1 + \cdots + i_n$.

Definition 2.4. *For each $I = (i_1, \cdots, i_n) \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ with $N \geq d\|I\|$, denote by $\mathcal{L}_N^I$ the set of all $\gamma \in \mathcal{K}[x_0, \ldots, x_M]_{N-d\|I\|}$ such that*

$$Q_1^{i_1} \cdots Q_n^{i_n}\gamma - \sum_{E=(e_1,\ldots,e_n)>I} Q_1^{e_1} \cdots Q_n^{e_n}\gamma_E \in \mathcal{I}_{\mathcal{K}}(V)_N.$$

for some $\gamma_E \in \mathcal{K}[x_0, \ldots, x_M]_{N-d\|E\|}$.

Denote by $\mathcal{L}^I$ the homogeneous ideal in $\mathcal{K}[x_0, \ldots, x_M]$ generated by $\cup_{N\geq d\|I\|}\mathcal{L}_N^I$.

Remark 2.5. *i) $\mathcal{L}_N^I$ is a $\mathcal{K}$-vector sub-space of $\mathcal{K}[x_0, \ldots, x_M]_{N-d\|I\|}$, and $(\mathcal{I}(V), Q_1, \ldots, Q_n)_{N-d\|I\|} \subset \mathcal{L}_N^I$, where $(\mathcal{I}(V), Q_1, \ldots, Q_n)$ is the ideal in $\mathcal{K}[x_0, \ldots, x_M]$ generated by $\mathcal{I}(V) \cup \{Q_1, \ldots, Q_n\}$.*

ii) For any $\gamma \in \mathcal{L}_N^I$ and $P \in \mathcal{K}[x_0, \ldots, x_M]_k$, we have $\gamma \cdot P \in \mathcal{L}_{N+k}^I$

iii) $\mathcal{L}^I \cap \mathcal{K}[x_0, \ldots, x_M]_{N-d\|I\|} = \mathcal{L}_N^I$.

iv) $\frac{\mathcal{K}[x_0,\ldots,x_M]}{\mathcal{L}^I}$ is a graded modul over the graded ring $\mathcal{K}[x_0, \ldots, x_M]$.

Set

$$m_N^I := \dim_{\mathcal{K}} \frac{\mathcal{K}[x_0, \ldots, x_M]_{N-d\|I\|}}{\mathcal{L}_N^I}.$$

For each positive integer N, denote by τ_N the set of all $I := (i_0, \ldots, i_n) \in \mathbb{N}_0^n$ with $N - d\|I\| \geq 0$. Let $\gamma_{I1}, \ldots, \gamma_{Im_N^I} \in \mathcal{K}[x_0, \ldots, x_M]_{N-d\|I\|}$ such that they form a basis of the $\mathcal{K}$-vector space $\frac{\mathcal{K}[x_0,\ldots,x_M]_{N-d\|I\|}}{\mathcal{L}_N^I}$.

Lemma 2.6. *$\{[Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_{I1}], \ldots, [Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_{Im_N^I}],\ I = (i_1, \ldots, i_n) \in \tau_N\}$ is a basis of the $\mathcal{K}$-vector space $\frac{\mathcal{K}[x_0,\ldots,x_M]_N}{\mathcal{I}_{\mathcal{K}}(V)_N}$.*

Proof. Firstly, we prove that:

$$\{[Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_{I1}], \ldots, [Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_{Im_N^I}],\ I = (i_1, \ldots, i_n) \in \tau_N\} \tag{2.3}$$

are linearly independent.
Indeed, let $t_{I\ell} \in \mathcal{K}$, $(I = (i_1, \ldots, i_n) \in \tau_N, \ell \in \{1, \ldots, m_N^I\})$ such that

$$\sum_{I \in \tau_N} \big(t_{I1}[Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_{I1}] + \cdots + t_{Im_N^I}[Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_{Im_N^I}] \big) = 0.$$

Then

$$\sum_{I \in \tau_N} Q_1^{i_1} \cdots Q_n^{i_n} \big(t_{I1}\gamma_{I1} + \cdots + t_{Im_N^I}\gamma_{Im_N^I} \big) \in \mathcal{I}_{\mathcal{K}}(V)_N. \tag{2.4}$$

By the definition of $\mathcal{L}_N^I$, and by (2.4), we get

$$t_{I^*1}\gamma_{I^*1} + \cdots + t_{I^*m_N^{I^*}}\gamma_{I^*m_N^{I^*}} \in \mathcal{L}_N^{I^*},$$

where I^* is the smallest element of τ_N.
On the other hand, $\{\gamma_{I^*1}, \ldots, \gamma_{I^*m_N^{I^*}}\}$ form a basis of $\frac{\mathcal{K}[x_0,\ldots,x_M]_{N-d\|I^*\|}}{\mathcal{L}_N^{I^*}}$.
Hence,

$$t_{I^*1} = \cdots = t_{I^*m_N^{I^*}} = 0. \tag{2.5}$$

Then, by (2.4), we have

$$\sum_{I \in \tau_N \setminus \{I^*\}} Q_1^{i_1} \cdots Q_n^{i_n} \big(t_{I1}\gamma_{I1} + \cdots + t_{Im_N^I}\gamma_{Im_N^I} \big) \in \mathcal{I}_{\mathcal{K}}(V)_N.$$

Then, similarly to (2.5), we have

$$t_{\tilde{I}1} = \cdots = t_{\tilde{I}m_N^{\tilde{I}}} = 0,$$

where $\tilde{I}$ is the smallest element of $\tau_N \setminus \{I^*\}$.
Continuing the above process, we get that $t_{I\ell} = 0$ for all $I \in \tau_N$ and $\ell \in \{1, \ldots, m_N^I\}$, and hence, we get (2.3).

Denote by $\mathcal{L}$ the $\mathcal{K}$-vector sub-space in $\mathcal{K}[x_0, \ldots, x_M]_N$ generated by

$$\{Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_{I1}, \ldots, Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_{Im_N^I},\ I = (i_1, \ldots, i_n) \in \tau_N\}.$$

Now we prove that: For any $I = (i_1, \ldots, i_n) \in \tau_N$, we have

$$Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_I \in \mathcal{L} + \mathcal{I}_{\mathcal{K}}(V)_N \tag{2.6}$$

for all $\gamma_I \in \mathcal{K}[x_0, \ldots, x_M]_{N-d\|I\|}$.

Set $I' = (i'_1, \ldots, i'_n) := \max\{I : I \in \tau_N\}$. Since $\gamma_{I'1}, \ldots, \gamma_{I'm_N^{I'}}$ form a basis of $\frac{\mathcal{K}[x_0,\ldots,x_M]_{N-d\|I'\|}}{\mathcal{L}_N^{I'}}$, for any $\gamma_{I'} \in \mathcal{K}[x_0, \ldots, x_M]_{N-d\|I'\|}$, we have

$$\gamma_{I'} = \sum_{\ell=1}^{m_N^{I'}} t_{I'\ell} \cdot \gamma_{I'\ell} + h_{I'\ell}, \text{ where } h_{I'\ell} \in \mathcal{L}_N^{I'}, \text{ and } t_{I'\ell} \in \mathcal{K}. \tag{2.7}$$

On the other hand, by the definition of $\mathcal{L}_N^{I'}$, we have $Q_1^{i'_1} \cdots Q_n^{i'_n} \cdot h_{I'\ell} \in \mathcal{I}_{\mathcal{K}}(V)_N$ (note that $I' = \max\{I : I \in \tau_N\}$). Hence,

$$Q_1^{i'_1} \cdots Q_n^{i'_n} \cdot \gamma_{I'} = \sum_{\ell=1}^{m_N^{I'}} t_{I'\ell} Q_1^{i'_1} \cdots Q_n^{i'_n} \cdot \gamma_{I'\ell} + Q_1^{i'_1} \cdots Q_n^{i'_n} \cdot h_{I'\ell} \in \mathcal{L} + \mathcal{I}_{\mathcal{K}}(V)_N.$$

We get (2.6) for the case where $I = I'$.

Assume that (2.6) holds for all $I > I^* = (i_1^*, \ldots, i_n^*)$. We prove that (2.6) holds also for $I = I^*$.
Indeed, similarly to (2.7), for any $\gamma_{I^*} \in \mathcal{K}[x_0, \ldots, x_M]_{N-d\|I^*\|}$, we have

$$\gamma_{I^*} = \sum_{\ell=1}^{m_N^{I^*}} t_{I^*\ell} \cdot \gamma_{I^*\ell} + h_{I^*\ell}, \text{ where } h_{I^*\ell} \in \mathcal{L}_N^{I^*}, \text{ and } t_{I^*\ell} \in \mathcal{K}.$$

Then,

$$Q_1^{i_1^*}\cdots Q_n^{i_n^*}\cdot\gamma_{I^*}=\sum_{\ell=1}^{m_N^{I^*}} t_{I^*\ell}Q_1^{i_1^*}\cdots Q_n^{i_n^*}\cdot\gamma_{I^*\ell}+Q_1^{i_1^*}\cdots Q_n^{i_n^*}\cdot h_{I^*\ell}. \tag{2.8}$$

Since $h_{I^*\ell}\in\mathcal{L}_N^{I^*}$, we have

$$Q_1^{i_1^*}\cdots Q_n^{i_n^*}\cdot h_{I^*\ell}-\sum_{E=(e_1,\ldots,e_n)>I^*} Q_1^{e_1}\cdots Q_n^{e_n}\cdot g_E\in\mathcal{I}_{\mathcal{K}}(V)_N,$$

for some $g_E\in\mathcal{K}[x_0,\ldots,x_M]_{N-d\|E\|}$.
Therefore, by the induction hypothesis,

$$Q_1^{i_1^*}\cdots Q_n^{i_n^*}\cdot h_{I^*\ell}\in\mathcal{L}+\mathcal{I}_{\mathcal{K}}(V)_N.$$

Then, by (2.8), we have

$$Q_1^{i_1^*}\cdots Q_n^{i_n^*}\cdot \gamma_{I^*}\in\mathcal{L}+\mathcal{I}_{\mathcal{K}}(V)_N.$$

This means that (2.6) holds for $I=I^*$. Hence, by (descending) induction we get (2.6).

For any $Q\in\mathcal{K}[x_0,\ldots,x_M]_N$, we write $Q=Q_1^0\cdots Q_n^0\cdot Q$. Then by (2.6), we have

$$Q\in\mathcal{L}+\mathcal{I}_{\mathcal{K}}(V)_N.$$

Hence,

$$\{[Q_1^{i_1}\cdots Q_n^{i_n}\cdot\gamma_{I1}],\ldots,[Q_1^{i_1}\cdots Q_n^{i_n}\cdot\gamma_{Im_N^I}],\ I=(i_1,\ldots,i_n)\in\tau_N\}$$

is a generating system of $\frac{\mathcal{K}[x_0,\ldots,x_M]_N}{\mathcal{I}_{\mathcal{K}}(V)_N}$. Combining with (2.3), we get the conclusion of Lemma 2.6. □

Lemma 2.7. $\#\{\mathcal{L}^I : I\in\mathbb{N}_0^n\}<\infty$.

Proof. Suppose that $\#\{\mathcal{L}^I : I\in\mathbb{N}_0^n\}=\infty$. Then there exists an infinite sequence $\{\mathcal{L}^{I_k}\}_{k=1}^\infty$ consisting of pairwise different ideals. We write $I_k=(i_{k1},\ldots,i_{kn})$. Since $i_{k\ell}\in\mathbb{N}_0$, there exists an infinite sequence of positve integers $p_1<p_2<p_3<\cdots$ such that $i_{p_1\ell}\leqslant i_{p_2\ell}\leqslant i_{p_3\ell}\leqslant\cdots$, for all $\ell=1,\ldots,n$: In fact, firstly we choose a sub-sequence $i_{q_11}\leqslant i_{q_21}\leqslant i_{q_31}\leqslant\cdots$ of $\{i_{k1}\}_{k=1}^\infty$. Next, we choose a sub-sequence of $i_{r_12}\leqslant i_{r_22}\leqslant i_{r_32}\leqslant\cdots$

of $\{i_{q_k2}\}_{k=1}^{\infty}$. Continuing the above process until obtaining a sub-sequence $i_{p_1n} \leqslant i_{p_2n} \leqslant i_{p_3n} \leqslant \cdots$.

We now prove that:

$$\mathcal{L}^{I_{p_1}} \subset \mathcal{L}^{I_{p_2}} \subset \mathcal{L}^{I_{p_3}} \subset \cdots. \tag{2.9}$$

Indeed, for any $\gamma \in \mathcal{L}_N^{I_{p_k}}$ (for any N and k satisfying $N - \|I_{p_k}\| \geq 0$), we have

$$Q_1^{i_{p_k1}} \cdots Q_n^{i_{p_kn}}\gamma - \sum_{E=(e_1,\ldots,e_n)>I_{p_k}} Q_1^{e_1} \cdots Q_n^{e_n}\gamma_E \in \mathcal{I}_{\mathcal{K}}(V)_N,$$

for some $\gamma_E \in \mathcal{K}[x_0, \ldots, x_M]_{N-d\|E\|}$.
Then, since $i_{p_{k+1}1} - i_{p_k1}, \ldots, i_{p_{k+1}n} - i_{p_kn}$ are non-negative integers, we have

$$Q_1^{i_{p_{k+1}1}} \cdots Q_n^{i_{p_{k+1}n}}\gamma - \sum_{E=(e_1,\ldots,e_n)>I_{p_k}} Q_1^{e_1+(i_{p_{k+1}1}-i_{p_k1})} \cdots Q_n^{e_n+(i_{p_{k+1}n}-i_{p_kn})}\gamma_E \in \mathcal{I}_{\mathcal{K}}(V)_N.$$

On the other hand since $E = (e_1, \ldots, e_n) > I_{p_k}$ we have $(e_1 + i_{p_{k+1}1} - i_{p_k1}, \ldots, e_n + i_{p_{k+1}n} - i_{p_kn}) > I_{p_{k+1}}$. Therefore, $\gamma \in \mathcal{L}^{I_{p_{k+1}}}_{N-d\|I_{p_k}\|+d\|I_{p_{k+1}}\|}$. Hence, $\mathcal{L}_N^{I_{p_k}} \subset \mathcal{L}^{I_{p_{k+1}}}_{N-d\|I_{p_k}\|+d\|I_{p_{k+1}}\|}$ for all k, N. Therefore, $\mathcal{L}^{I_{p_k}} \subset \mathcal{L}^{I_{p_{k+1}}}$ for all k. We get (2.9).
Since $\mathcal{K}[x_0, \ldots, x_M]$ is a noetherian ring, the chain of ideals in (2.9) becomes finally stationary. This is a contradiction. □

Lemma 2.8. *There are integers n_0, c and c' such that the following assertions hold.*

i) $\dim_{\mathcal{K}} \frac{\mathcal{K}[x_0,\ldots,x_M]_{N-d\|I\|}}{(\mathcal{I}(V),Q_1,\ldots,Q_n)_{N-d\|I\|}} = c$ *for all $I \in \mathbb{N}_0^n, N \in \mathbb{N}_0$ satisfying $N - d\|I\| \geq n_0$.*

ii) For each $I \in \mathbb{N}_0^n$ there is an integer m^I such that $m^I = m_N^I$ for all $N \in \mathbb{N}_0$ satisfying $N - d\|I\| \geq n_0$.

iii) $m_N^I \leqslant c'$, for all $I \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ satisfying $N - d \cdot \|I\| \geq 0$.

Proof. For each z in $\mathbb{C}$ such that all coefficients of Q_j $(j = 1, \ldots, n\}$ are holomorphic at z, we denote by $(\mathcal{I}(V), Q(z), \ldots, Q(z))$ the ideal in $\mathbb{C}[x_0, \ldots, x_M]$ generated by $\mathcal{I}(V) \cup \{Q_1(z), \ldots, Q_n(z)\}$.

We have

$$(\mathcal{I}(V), Q_1(z), \ldots, Q_n(z)) \subset (\mathcal{I}(V), Q_1, \ldots, Q_n)(z). \tag{2.10}$$

Indeed, for any $P \in (\mathcal{I}(V), Q_1(z), \dots, Q_n(z))$, we write $P = G + Q_1(z) \cdot P_1 + \cdots + Q_n(z) \cdot P_n$, where $G \in \mathcal{I}(V)$, and $P_i \in \mathbb{C}[x_0, \dots, x_M]$. Take $\widetilde{P} := G + Q_1 \cdot P_1 + \cdots + Q_n \cdot P_n \in (\mathcal{I}(V), Q_1, \dots, Q_n)$, then all coefficients of $\widetilde{P}$ are holomorphic at z. It is clear that $\widetilde{P}(z) = P$. Hence, we get (2.10).

Let N be an arbitrary positive integer and I be an arbitrary element in τ_N. Let $\{h_k := \sum_{j=1}^n Q_j \cdot R_{jk} + \sum_{j=1}^{m_k} \gamma_{jk} \cdot g_{jk}\}_{k=1}^K$ be a basic system of $(\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\cdot\|I\|}$, where $g_{jk} \in \mathcal{I}(V)$, and $R_{jk}, \gamma_{jk}, \in \mathcal{K}[x_0, \dots, x_M]$ satisfying $\deg(Q_j \cdot R_{jk}) = \deg(\gamma_{jk} \cdot g_{jk}) = N - d \cdot \|I\|$. By Lemma 2.2, and since $\{Q_0, \dots, Q_n\}$ is a $V-$ admissible set, there exists $a \in \mathbb{C}$ such that:

i) $\{h_k(a)\}_{k=1}^K$ is a basic system of $(\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\cdot\|I\|}(a)$,

ii) all coefficients of $Q_j, R_{jk}, \gamma_{jk}, g_{jk}$ are holomorphic at a, and

iii) the homogeneous polynomials $Q_0(a), \dots, Q_n(a) \in \mathbb{C}[x_0, \dots, x_M]$ have no common zero points in V.

On the other hand, it is clear that $h_k(a) \in (\mathcal{I}(V), Q_1(a), \dots, Q_n(a))$, for all $k = 1, \dots, K$. Hence, by (2.10), and by i), we have

$$(\mathcal{I}(V), Q_1(a), \dots, Q_n(a))_{N-d\cdot\|I\|} = (\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\cdot\|I\|}(a).$$

Then, we have

$$\begin{aligned}\dim_{\mathcal{K}}(\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\cdot\|I\|} = K &= \dim_{\mathbb{C}}(\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\cdot\|I\|}(a) \\ &= \dim_{\mathbb{C}}(\mathcal{I}(V), Q_1(a), \dots, Q_n(a))_{N-d\cdot\|I\|}.\end{aligned}$$

Therefore,

$$\dim_{\mathcal{K}} \frac{\mathcal{K}[x_0, \dots, x_M]_{N-d\|I\|}}{(\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\|I\|}} = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_0, \dots, x_M]_{N-d\|I\|}}{(\mathcal{I}(V), Q_1(a), \dots, Q_n(a))_{N-d\|I\|}}. \tag{2.11}$$

On the other hand, by the Hilbert-Serre Theorem ([12], Theorem 7.5), there exist positive integers n_1, c such that

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[x_0, \dots, x_M]_{N-d\|I\|}}{(\mathcal{I}(V), Q_1(a), \dots, Q_n(a))_{N-d\|I\|}} = c,$$

for all $I \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ satisfying $N - d\|I\| \geq n_1$.

Combining with (2.11), we have

$$\dim_{\mathcal{K}} \frac{\mathcal{K}[x_0, \dots, x_M]_{N-d\|I\|}}{(\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\|I\|}} = c, \tag{2.12}$$

for all $I \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ satisfying $N - d\|I\| \geq n_1$.

Let h^I and h be the Hilbert functions of $\frac{\mathcal{K}[x_0,\ldots,x_M]}{\mathcal{L}^I}$ and $\frac{\mathcal{K}[x_0,\ldots,x_M]}{(\mathcal{I}(V),Q_1,\ldots,Q_n)}$, respectively. Since $(\mathcal{I}(V), Q_1, \ldots, Q_n) \subset \mathcal{L}^I$, we have $h^I \leqslant h$. On the other hand, by Matsumura [17], Theorem 14, $h^I(k)$ is a polynomial in k for all $k >> 0$ and by (2.12), we have $h(k) = c$ for all $k \geq n_1$. Hence, there are constants m^I, n_2 such that $h^I(k) = m^I$ for all $k \geq n_2$ and then $m_N^I = h^I(N - d\|I\|) = m^I$ for all $N \in \mathbb{N}_0$ satisfying $N - d\|I\| \geq n_2$. By Lemma 2.7, we may choose n_2 common for all I. Taking $n_0 := \max\{n_1, n_2\}$, we get Lemma 2.8, i) and ii).

We have $m_N^I = h^I(N - d\|I\|) \leqslant h(N - d\|I\|) \leqslant \max\{c, h(k) : k = 0, \ldots, n_0\}$. Hence, taking $c' := \max\{c, h(k) : k = 0, \ldots, n_0\}$, we get Lemma 2.8 iii).

□

Set

$$m := \min_{I \in \mathbb{N}_0^n} m^I.$$

We fix $I_0 = (i_{01}, \ldots, i_{0n}) \in \mathbb{N}_0^n$, and $N_0 \in \mathbb{N}_0$ such that $N_0 - d\|I_0\| \geq n_0$ and $m_{N_0}^{I_0} = m$.

For each positive integer N, divisible by d, denote by τ_N^0 the set of all $I = (i_1, \ldots, i_n) \in \tau_N$ such that $N - d\|I\| \geq n_0$ and $i_k \geq \max\{i_{01}, \ldots, i_{0n}\}$, for all $k \in \{1, \ldots, n\}$.
We have

$$\#\tau_N = \binom{\frac{N}{d} + n}{n} = \frac{1}{d^n} \cdot \frac{N^n}{n!} + O(N^{n-1}),$$

$$\#\{I \in \tau_N : N - d\|I\| \leqslant n_0\} = O(N^{n-1}),$$

$$\#\{I = (i_1, \ldots, i_n) \in \tau_N : \ i_k < \max_{1 \leqslant \ell \leqslant n} i_{0\ell}, \text{ for some } k\} = O(N^{n-1}), \text{ and so}$$

$$\#\tau_N^0 = \frac{1}{d^n} \cdot \frac{N^n}{n!} + O(N^{n-1}). \tag{2.13}$$

Lemma 2.9. $m_N^I = \deg V \cdot d^n$ *for all* $N >> 0$, *divisible by* d, *and* $I \in \tau_N^0$.

Proof. For any $\gamma \in \mathcal{L}_{N_0}^{I^0}$, we have

$$T := Q_1^{i_{01}} \cdots Q_n^{i_{0n}} \gamma - \sum_{E = (e_1, \ldots, e_n) > I_0} Q_1^{e_1} \cdots Q_n^{e_n} \gamma_E \in \mathcal{I}_{\mathcal{K}}(V)_{N_0},$$

for some $\gamma_E \in \mathcal{K}[x_0, \ldots, x_M]_{N-d\|E\|}$.
Then, for any $I = (i_1, \ldots, i_n) \in \tau_N^0$, we have

$$\begin{aligned} Q_1^{i_1} \cdots Q_n^{i_n} \gamma - \sum_{E=(e_1,\ldots,e_n)>I_0} Q_1^{e_1+i_1-i_{01}} \cdots Q_n^{e_n+i_n-i_{0n}} \gamma_E \\ = Q_1^{i_1-i_{01}} \cdots Q_n^{i_n-i_{0n}} \cdot T \in \mathcal{I}_{\mathcal{K}}(V)_{N_0}. \end{aligned} \tag{2.14}$$

On the other hand since $I \in \tau_N^0$ and $E > I_0$, we have $(e_1 + i_1 - i_{01}, \ldots, e_n + i_n - i_{0n}) > I$.
Hence, by (2.14) we have

$$\gamma \in \mathcal{L}^I_{N_0+d\|I\|-d\|I_0\|}.$$

This implies that

$$\mathcal{L}^{I_0}_{N_0} \subset \mathcal{L}^I_{N_0+d\|I\|-d\|I_0\|}.$$

Then

$$\begin{aligned} m = m^{I_0}_{N_0} &= \dim_{\mathcal{K}} \frac{\mathcal{K}[x_0, \ldots, x_M]_{N_0-d\|I_0\|}}{\mathcal{L}^{I_0}_{N_0}} \\ &\geq \dim_{\mathcal{K}} \frac{\mathcal{K}[x_0, \ldots, x_M]_{N_0-d\|I_0\|}}{\mathcal{L}^I_{N_0+d\|I\|-d\|I_0\|}} \\ &= m^I_{N_0+d\|I\|-d\|I_0\|}. \end{aligned} \tag{2.15}$$

On the other hand since $(N_0 + d\|I\| - d\|I_0\|) - d\|I\| = N_0 - d\|I_0\| \geq n_0$, and $N - \|I\| \geq n_0$ (note that $I \in \tau_N^0$), by Lemma 2.8, we have

$$m^I = m^I_{N_0+d\|I\|-d\|I_0\|} = m^I_N.$$

Hence, by (2.15), $m \geq m^I = m^I_N$. Then, by the minimum property of m, we get that

$$m^I_N = m \text{ for all } I \in \tau_N^0. \tag{2.16}$$

We now prove that:

$$\dim_{\mathcal{K}} \mathcal{I}_{\mathcal{K}}(V)_N = \dim_{\mathbb{C}} \mathcal{I}(V)_N. \tag{2.17}$$

Indeed, let $\{P_1, \ldots, P_s\}$ be a basis of the $\mathbb{C}-$ vector space $\mathcal{I}(V)_N$. It is clear that $\mathcal{I}_{\mathcal{K}}(V)_N$ is a vector space over $\mathcal{K}$ generated by $\mathcal{I}(V)_N$, therefore

$\{P_1, \dots, P_s\}$ is also a generating system of $\mathcal{I}_{\mathcal{K}}(V)_N$. Then, for (2.17), it suffices to prove that if $t_1, \dots, t_s \in \mathcal{K}$ satisfy

$$t_1 \cdot P_1 + \cdots + t_s \cdot P_s \equiv 0, \tag{2.18}$$

then $t_1 = \cdots = t_s \equiv 0$. We rewrite (2.18) in the following form

$$A \cdot \begin{pmatrix} t_1 \\ \cdot \\ \cdot \\ \cdot \\ t_s \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix},$$

where $A \in$Mat$(\binom{M+N}{N} \times s, \mathcal{K})$.
If the above system of linear equations has non-trivial solutions, then $\text{rank}_{\mathcal{K}} A < s$. Then $\text{rank}_{\mathbb{C}} A(z) < s$ for all $z \in \mathbb{C}$ excluding a discrete set. Take $a \in \mathbb{C}$ such that $\text{rank}_{\mathbb{C}} A(a) < s$. Then the following system of linear equations

$$A(a) \cdot \begin{pmatrix} t_1 \\ \cdot \\ \cdot \\ \cdot \\ t_s \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix},$$

has some non-trivial solution $(t_1, \dots, t_s) = (\alpha_1, \dots, \alpha_s) \in \mathbb{C}^s \setminus \{0\}$. Then $\alpha_1 \cdot P_1 + \cdots + \alpha_s \cdot P_s \equiv 0$, this is a contradiction. Hence, we get (2.17).

By Lemma 2.6 and (2.17), we have

$$\begin{aligned} \sum_{I \in \tau_N} m_N^I = \dim_{\mathcal{K}} \frac{\mathcal{K}[x_0, \dots, x_M]_N}{\mathcal{I}_{\mathcal{K}}(V)_N} &= \dim_{\mathbb{C}} \frac{\mathbb{C}[x_0, \dots, x_M]_N}{\mathcal{I}(V)_N} \\ &= \deg V \cdot \frac{N^n}{n!} + O(N^{n-1}), \end{aligned} \tag{2.19}$$

for all N large enough.
Combining with (2.16), we have

$$m \cdot \#\tau_N^0 + \sum_{I \in \tau_N \setminus \tau_N^0} m_N^I = \deg V \cdot \frac{N^n}{n!} + O(N^{n-1}). \tag{2.20}$$

On the other hand by Lemma 2.8, $m_N^I \leqslant c'$, for all $I \in \tau_N \setminus \tau_N^0$. Hence, by (2.13), we have

$$m = \deg V \cdot d^n.$$

Combining with (2.16), we have

$$m_N^I = \deg V \cdot d^n$$

for all $I \in \tau_N^0$. □

Lemma 2.10. *For each $s \in \{1, \dots, n\}$, and for $N >> 0$, divisible by d, we have:*

$$\sum_{I=(i_1,\dots,i_n)\in\tau_N} m_N^I \cdot i_s \geq \frac{\deg V}{d \cdot (n+1)!} N^{n+1} - O(N^n).$$

Proof. Firstly, we note that if $I = (i_1, \dots, i_n) \in \tau_N^0$, then all symmetry $I' = (i_{\sigma(1)}, \dots, i_{\sigma(n)})$ of I also belongs to τ_N^0. On the other hand, by Lemma 2.9, we have $m_N^I = \deg V \cdot d^n$, for all $I \in \tau_N^0$. Therefore, by (2.13) we have

$$\begin{aligned}
\sum_{I=(i_1,\dots,i_n)\in\tau_N^0} m_N^I \cdot i_1 = \cdots = & \sum_{I=(i_1,\dots,i_n)\in\tau_N^0} m_N^I \cdot i_n \\
&= \deg V \cdot d^n \cdot \sum_{I\in\tau_N^0} \frac{\|I\|}{n} \\
&= \deg V \cdot d^n \cdot \left(\sum_{I\in\tau_N} \frac{\|I\|}{n} - \sum_{I\in\tau_N\setminus\tau_N^0} \frac{\|I\|}{n} \right) \\
&\geq \deg V \cdot d^n \left(\sum_{k=0}^{\frac{N}{d}} \frac{k}{n} \cdot \binom{k+n-1}{n-1} - \left(\#\tau_N - \#\tau_N^0\right) \cdot \frac{N}{nd} \right) \\
&= \deg V \cdot d^n \left(\sum_{k=0}^{\frac{N}{d}} \frac{k}{n} \cdot \binom{k+n-1}{n-1} - O(N^{n-1}) \cdot \frac{N}{nd} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \deg V \cdot d^n \sum_{k=1}^{\frac{N}{d}} \binom{k+n-1}{n} - O(N^n) \\
&= \deg V \cdot d^n \binom{\frac{N}{d}+n}{n+1} - O(N^n) \\
&\geq \frac{\deg V}{d\cdot (n+1)!} N^{n+1} - O(N^n).
\end{aligned}
$$

Hence, for each $i \in \{1, \ldots, n\}$

$$
\begin{aligned}
\sum_{I=(i_1,\ldots,i_n)\in\tau_N} m_N^I \cdot i_s &\geq \sum_{I=(i_1,\ldots,i_n)\in\tau_N^0} m_N^I \cdot i_s \\
&\geq \frac{\deg V}{d\cdot (n+1)!} N^{n+1} - O(N^n).
\end{aligned}
$$

□

We recall that by (2.19), for $N >> 0$, we have

$$
\dim_{\mathcal{K}} \frac{\mathcal{K}[x_0, \ldots, x_M]_N}{\mathcal{I}_{\mathcal{K}}(V)_N} = H_V(N) = \deg V \cdot \frac{N^n}{n!} + O(N^{n-1}).
$$

Therefore, from Lemmas 2.6, 2.10 we get immediately the following result.

Lemma 2.11. *For all $N >> 0$ divisible by d, there are homogeneous polynomials $\phi_1, \ldots, \phi_{H_V(N)}$ in $\mathcal{K}[x_0, \ldots, x_M]_N$ such that they form a basis of the $\mathcal{K}-$ vector space $\frac{\mathcal{K}[x_0,\ldots,x_M]_N}{\mathcal{I}_{\mathcal{K}}(V)_N}$, and*

$$
\prod_{j=1}^{H_V(N)} \phi_j - \big(Q_1 \cdots Q_n\big)^{\frac{\deg V\cdot N^{n+1}}{d\cdot(n+1)!} - u(N)} \cdot P \in \mathcal{I}_{\mathcal{K}}(V)_{N\cdot H_V(N)}\,,
$$

where $u(N)$ is a function in N satisfying $u(N) \leqslant O(N^n)$, P is a homogeneous polynomial of degree

$$
N\cdot H_V(N) - \frac{n\cdot \deg V \cdot N^{n+1}}{(n+1)!} + nd\cdot u(N) = \frac{\deg V\cdot N^{n+1}}{(n+1)!} + O(N^n).
$$

Lemma 2.12 (see [18])**.** *Let f be a non-constant holomorphic map of $\mathbb{C}$ into $\mathbb{CP}^M$. Let $H_j = a_{j0}x_0 + \cdots + a_{jM}x_M, j \in \{1, \ldots, q\}$ be q linear homogeneous* *polynomials in* $\mathcal{K}_f[x_0,\dots,x_M]$. *Denote by* $\mathcal{K}_{\{H_j\}_{j=1}^q}$ *the field over* $\mathbb{C}$ *of all meromorphic functions on* $\mathbb{C}$ *generated by* $\{a_{ji}, i=0,\dots,M; j=1,\dots,q\}$. *Assume that* f *is linearly non-degenerate over* $\mathcal{K}_{\{H_j\}_{j=1}^q}$. *Then for each* $\epsilon>0$, *we have*

$$\Big\|\frac{1}{2\pi}\int_0^{2\pi}\max_K\log\prod_{k\in K}\Big(\frac{\|f\|\cdot\max_{i=0,\dots,M}|a_{ki}|}{|H_k(f)|}(re^{i\theta})\Big)d\theta\leqslant(M+1+\epsilon)T_f(r),\tag{2.21}$$

where $\max_K$ *is taken over all subsets* $K\subset\{1,\dots,q\}$ *such that the polynomials* H_j, $j\in K$ *are linearly independent over* $\mathcal{K}_{\{H_j\}_{j=1}^q}$.

Remark 2.13. *Since the coefficients of the* $H_j's$ *are small functions (with respect to* f*), by the First Main Theorem, and by (2.21), for each* $\epsilon>0$, *we have*

$$\Big\|\frac{1}{2\pi}\int_0^{2\pi}\max_K\log\prod_{k\in K}\Big(\frac{\|f\|}{|H_k(f)|}(re^{i\theta})\Big)d\theta\leqslant(M+1+\epsilon)T_f(r).$$

3 Proof of the Main Theorem

Replacing Q_j by $Q^{\frac{d}{d_j}}$, where d is the l.c.m of the Q_j's, we may assume that the polynomials $Q_1,\dots,Q_q$ have the same degree d. Let $N>>0$ be an integer divisible by d. For each $J:=\{j_1,\dots,j_n\}\subset\{1,\dots,q\}$, by Lemma 2.11 (for $\mathcal{K}:=\mathcal{K}_{\mathcal{Q}}$), there are homogeneous polynomials $\phi_1^J,\dots,\phi_{H_V(N)}^J$ (depending on J) in $\mathcal{K}_{\mathcal{Q}}[x_0,\dots,x_M]$ and there are functions (common for all J) $u(N),v(N)\leqslant O(N^n)$ such that they form a basis of the $\mathcal{K}_{\mathcal{Q}}-$ vector space $\frac{\mathcal{K}_{\mathcal{Q}}[x_0,\dots,x_M]}{\mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)_N}$, and

$$\prod_{\ell=1}^{H_V(N)}\phi_\ell^J-\big(Q_{j_1}\cdots Q_{j_n}\big)^{\frac{\deg V\cdot N^{n+1}}{d\cdot(n+1)!}-u(N)}\cdot P_J\in\mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)_{N\cdot H_V(N)}\,,$$

where P_J is a homogeneous polynomial of degree $\frac{\deg V\cdot N^{n+1}}{(n+1)!}+v(N)$.
On the other hand, for any $Q\in\mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)_{N\cdot H_V(N)}$, we have $Q(f)\equiv 0$.
Therefore

$$\prod_{\ell=1}^{H_V(N)}\phi_\ell^J(f)=\big(Q_{j_1}(f)\cdots Q_{j_n}(f)\big)^{\frac{\deg V\cdot N^{n+1}}{d\cdot(n+1)!}-u(N)}\cdot P_J(f).$$

Since the coefficients of P_J are small functions (with respect to f), it is easy to see that there exist $h_J \in \mathcal{C}_f$ such that

$$|P_J(f)| \leqslant \|f\|^{\deg P_J} \cdot h_J = \|f\|^{\frac{\deg V \cdot N^{n+1}}{(n+1)!} + v(N)} \cdot h_J.$$

Hence,

$$\log\Big(\prod_{\ell=1}^{H_V(N)} |\phi_\ell^J(f)|\Big) \leqslant \Big(\frac{\deg V \cdot N^{n+1}}{d \cdot (n+1)!} - u(N)\Big) \cdot \log\big|Q_{j_1}(f) \cdots Q_{j_n}(f)\big| + \log^+ h_J$$
$$+ \big(\frac{\deg V \cdot N^{n+1}}{(n+1)!} + v(N)\big) \cdot \log \|f\|.$$

This implies that there are functions $\omega_1(N), \omega_2(N) \leqslant O(\frac{1}{N})$ such that

$$\log\big(|Q_{j_1}(f)| \cdots |Q_{j_n}(f)|\big) \geq \Big(\frac{d \cdot (n+1)!}{\deg V \cdot N^{n+1}} - \frac{\omega_1(N)}{N^{n+1}}\Big) \cdot \log\Big(\prod_{\ell=1}^{H_V(N)} |\phi_\ell^J(f)|\Big)$$
$$-\frac{1}{N^{n+1}} \log^+ \widetilde{h}_J - \big(d + \omega_2(N)\big) \cdot \log \|f\|, \qquad (3.1)$$

for some $\widetilde{h}_J \in \mathcal{C}_f$.

We fix homogeneous polynomials $\Phi_1, \ldots, \Phi_{H_V(N)} \in \mathcal{K}_{\mathcal{Q}}[x_0, \ldots, x_M]_N$ such that they form a basis of the $\mathcal{K}_{\mathcal{Q}}-$ vector space $\frac{\mathcal{K}_{\mathcal{Q}}[x_0,\ldots,x_M]_N}{\mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)_N}$. Then for each subset $J := \{j_1, \ldots, j_n\} \in \{1, \ldots, q\}$, there exist homogeneous linear polynomials $L_1^J, \ldots L_{H_V(N)}^J \in \mathcal{K}_{\mathcal{Q}}[y_1, \ldots, y_{H_V(N)}]$ such that they are linearly independent over $\mathcal{K}_{\mathcal{Q}}$ and

$$\phi_\ell^J - L_\ell^J(\Phi_1, \ldots, \Phi_{H_V(N)}) \in \mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)_N, \text{ for all } \ell \in \{1, \ldots, H_V(N)\}. \qquad (3.2)$$

It is easy to see that there exists a meromorphic function φ such that $N_\varphi(r) = o(T_f(r))$, $N_{\frac{1}{\varphi}}(r) = o(T_f(r))$ and $\frac{\Phi_1(f)}{\varphi}, \ldots, \frac{\Phi_{H_V(N)}(f)}{\varphi}$ are holomorphic and have no common zeros (note that all coefficients of Φ_ℓ are in $\mathcal{K}_{\mathcal{Q}} \subset \mathcal{K}_f$). Let $F : \mathbb{C} \to \mathbb{CP}^{H_V(N)-1}$ be the holomorphic map with the reduced representation $F := \big(\frac{\Phi_1(f)}{\varphi} : \cdots : \frac{\Phi_{H_V(N)}(f)}{\varphi}\big)$. Since f is algebraically nondegenerate over $\mathcal{K}_{\mathcal{Q}}$, and since the polynomials $\Phi_1, \ldots, \Phi_{H_V(N)}$ form a basis of $\frac{\mathcal{K}_{\mathcal{Q}}[x_0,\ldots,x_M]_N}{\mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)_N}$, we get that F is linearly non-degenerate over $\mathcal{K}_{\mathcal{Q}}$. As a corollary,

F is linearly non-degenerate over the field over $\mathbb{C}$ generated by all coefficients of L_ℓ's.

It is easy to see that

$$T_F(r) \leqslant N \cdot T_f(r) + o(T_f(r)). \tag{3.3}$$

In order to simplify the writing of the following series of inequalities, put $A(N) := \left(\frac{d\cdot(n+1)!}{\deg V\cdot N^{n+1}} - \frac{\omega_1(N)}{N^{n+1}}\right)$. By (3.2), for all $\ell \in \{1, \ldots, H_V(N)\}$ we have

$$\log|\phi_\ell^J(f)| = \log|L_\ell^J(F)| + \log|\varphi|.$$

Hence, by (3.1), and by taking $\widetilde{h} \in \mathcal{C}_f$ such that $\log^+ \widetilde{h}_J \leqslant \log^+ \widetilde{h}$ for all J, we get

$$\begin{aligned}
\log\big(|Q_{j_1}(f)|\cdots|Q_{j_n}(f)|\big) \geq A(N)\cdot\Big(H_V(N)\cdot\log|\varphi| + \log\Big(\prod_{\ell=1}^{H_V(N)}|L_\ell^J(F)|\Big)\Big)\\
- \frac{1}{N^{n+1}}\log^+\widetilde{h}_J - \big(d+\omega_2(N)\big)\log\|f\|\\
\geq A(N)\cdot\log\Big(\prod_{\ell=1}^{H_V(N)}|L_\ell^J(F)|\Big) + A(N)\cdot H_V(N)\cdot\log|\varphi|\\
- \log^+\widetilde{h} - \big(d+\omega_2(N)\big)\log\|f\|.
\end{aligned} \tag{3.4}$$

Then, by Lemma 2.3, and by increasing $\widetilde{h} \in \mathcal{C}_f$ if necessary, we get

$$\begin{aligned}
\log\prod_{j=1}^{q}|Q_j(f)| &= \max_{\{\beta_1,\ldots,\beta_{q-n}\}\subset\{1,\ldots,q\}}\log|Q_{\beta_1}(f)\cdots Q_{\beta_{q-n}}(f)|\\
&\quad+ \min_{J=\{j_1,\ldots,j_n\}\subset\{1,\ldots,q\}}\log|Q_{j_1}(f)\cdots Q_{j_n}(f)|\\
&\geq (q-n)d\cdot\log\|f\| + \min_{J\subset\{1,\ldots,q\},\#J=n} A(N)\cdot\log\Big(\prod_{\ell=1}^{H_V(N)}|L_\ell^J(F)|\Big)\\
&\quad- \big(d+\omega_2(N)\big)\log\|f\| + A(N)\cdot H_V(N)\cdot\log|\varphi| - \log^+\widetilde{h}\\
&= (q-n-1)d\cdot\log\|f\| + \min_{J\subset\{1,\ldots,q\},\#J=n} A(N)\cdot\log\Big(\prod_{\ell=1}^{H_V(N)}|L_\ell^J(F)|\Big)\\
&\quad- \omega_2(N)\cdot\log\|f\| + A(N)\cdot H_V(N)\cdot\log|\varphi| - \log^+\widetilde{h}
\end{aligned} \tag{3.5}$$

Now for given $\epsilon > 0$ we fix $N = N(\epsilon)$ big enough such that

$$\omega_2(N) \leqslant \frac{\epsilon}{3} \text{ and } A(N) < 1\,. \tag{3.6}$$

By using Remark 2.13 to the holomorphic map $F : \mathbb{C} \to \mathbb{CP}^{H_V(N)-1}$, the error constant $\frac{\epsilon}{2N} > 0$ and the system of linear polynomials $L_1^J, \ldots L_{H_V(N)}^J \in \mathcal{K}_{\mathcal{Q}}[y_1, \ldots, y_{H_V(N)}]$, where J runs over all subsets $J := \{j_1, \ldots, j_n\} \in \{1, \ldots, q\}$, we get:

$$\begin{aligned}
&\Big\| \frac{1}{2\pi} \int_0^{2\pi} \max_{J \subset \{1,\ldots,q\}, \# J = n} \log \Big(\prod_{\ell=1}^{H_V(N)} \frac{\|F\|}{|L_\ell^J(F)|}(re^{i\theta}) \Big) d\theta \\
\leqslant \frac{1}{2\pi} \int_0^{2\pi} &\max_K \log \prod_{k \in K} \Big(\frac{\|F\|}{|L_\ell^J(F)|}(re^{i\theta}) \Big) d\theta \leqslant (H_V(N) + \frac{\epsilon}{2N}) T_F(r)\,,
\end{aligned} \tag{3.7}$$

where $\max_K$ is taken over all subsets of the system of linear polynomials $L_1^J, \ldots L_{H_V(N)}^J \in \mathcal{K}_{\mathcal{Q}}[y_1, \ldots, y_{H_V(N)}]$, where J runs over all subsets $J := \{j_1, \ldots, j_n\} \in \{1, \ldots, q\}$, such that these linear polynomials are linearly independent over $\mathcal{K}_{\mathcal{Q}}$.

So, by integrating (3.5) and combining with (3.6) and (3.7) we have (using that $N_\varphi(r) = o(T_f(r))$, $N_{\frac{1}{\varphi}}(r) = o(T_f(r))$, that $A(N) \cdot H_V(N) \leqslant O(\frac{1}{N})$ and that $\widetilde{h} \in \mathcal{C}_f$)

$$\begin{aligned}
\Big\| \sum_{j=1}^q N_f(r, Q_j) \geq & d(q-n-1) T_f(r) - \frac{\epsilon}{3} T_f(r) + A(N) \cdot H_V(N) \cdot \Big(N_\varphi(r) - N_{\frac{1}{\varphi}}(r) \Big) - \frac{\epsilon}{12} T_f(r) \\
& + A(N) \cdot \frac{1}{2\pi} \int_0^{2\pi} \min_{J \subset \{1,\ldots,q\}, \# J = n} \log \Big(\prod_{\ell=1}^{H_V(N)} |L_\ell^J(F)|(re^{i\theta}) \Big) d\theta \\
\geq & d(q-n-1) T_f(r) - \frac{\epsilon}{3} T_f(r) - \frac{\epsilon}{12} T_f(r) - \frac{\epsilon}{12} T_f(r) \\
& + A(N) \cdot \frac{1}{2\pi} \int_0^{2\pi} \min_{J \subset \{1,\ldots,q\}, \# J = n} \log \Big(\prod_{\ell=1}^{H_V(N)} |L_\ell^J(F)|(re^{i\theta}) \Big) d\theta
\end{aligned}$$

$$
\begin{aligned}
=&d(q-n-1)T_f(r)-\frac{\epsilon}{2}T_f(r)\\
&-A(N)\cdot\frac{1}{2\pi}\int_0^{2\pi}\max_{J\subset\{1,\dots,q\},\#J=n}\log\Big(\prod_{\ell=1}^{H_V(N)}\frac{\|F\|}{|L_\ell^J(F)|}(re^{i\theta})\Big)d\theta\\
&+A(N)\cdot H_V(N)\cdot T_F(r)\\
\geq&d(q-n-1)T_f(r)-A(N)\big(H_V(N)+\frac{\epsilon}{2N}\big)T_F(r)\\
&+A(N)\cdot H_V(N)\cdot T_F(r)-\frac{\epsilon}{2}T_f(r)\\
\geq&d(q-n-1)T_f(r)-\frac{\epsilon}{2N}T_F(r)-\frac{\epsilon}{2}T_f(r)\\
\geq&d(q-n-1-\epsilon)T_f(r).
\end{aligned}
$$

This completes the proof of the Main Theorem. □

References

[1] D. P. An, S. D. Quang and D. D. Thai, *The second main theorem for meromorphic mappings into a complex projective space*, Acta Math. Vietnam **38** (2013), 187-205.

[2] H. Cartan, *Sur les zéros des combinaisons linéaires de p fonctions holomorphes données*, Mathematica **7** (1933), 5-31.

[3] Z. Chen, M. Ru and Q. Yan, *The truncated second main theorem and uniqueness theorems*, Science China **53** (2010), 605-616.

[4] Z. Chen, M. Ru and Q. Yan, *The degenerated second main theorem and Schmidt's subspace theorem*, Science China **55** (2012), 1367-1380.

[5] Z. Chen, M. Ru and Q. Yan, *Schmidt's subspace theorem with moving hypersurfaces*, Int. Math. Res. Notices **2015** (2015), 6305-6329.

[6] P. Corvaja and U. Zannier, *On a general Thue's equation*, Amer. J. Math. **126** (2004), 1033-1055.

[7] G. Dethloff and T. V. Tan, *A second main theorem for moving hypersurface targets*, Houston J. Math. **37** (2011), 79-111.

[8] G. Dethloff, T. V. Tan and D. D. Thai, *An extension of the Cartan-Nochka second main theorem for hypersurfaces*, Int. J. Math. **22** (2011), 863-885.

[9] A. E. Eremenko and M. L. Sodin, *The value distribution of meromorphic functions and meromorphic curves from the point of view of potential theory*, St. Petersburg Math. J. **3** (1992), 109-136.

[10] J. H. Evertse and R. G. Ferretti, *Diophantine inequalities on projective varieties*, Internat. Math. Res. Notices **25** (2002), 1295-1330.

[11] J. H. Evertse and R. G. Ferretti, *A generalization of the subspace theorem with polynomials of higher degree*, Developments in Mathematics **16** (2008), 175-198, Springer-Verlag, New York.

[12] R. Hartshorne, Algebraic Geometry, Grad. Texts in Math. vol 52, Springer-Verlag, New York, 1977.

[13] Q. Ji, Q. Yan, and G. Yu, *Holomorphic curves into algebraic varieties intersecting divisors in subgeneral position,* Math. Ann. (2018). https://doi.org/10.1007/s00208-018-1661-4.

[14] G. Le, *Schmidt's subspace theorem with moving hypersurfaces*, Int. J. Number Theory **11** (2015), 139-158.

[15] G. Le, *An explicit estimate on multiplicity truncation in the degenerated Second Main Theorem*, Houston J. Math. **42** (2016), 447-462.

[16] A. Levin, *On a Schmidt subspace theorem for algebraic points*, Duke Math. J. **163** (2014), 2841-2885.

[17] H Matsumura, Commutative Algebra, Benjamin/Cummings Publication Company, Massachusetts, 1980.

[18] M. Ru, *On a general form of the Second Main Theorem*, Trans. AMS **349** (1997), 5093-5105.

[19] M. Ru and P. Vojta *Schmidt's subspace theorem with moving targets*, Invent. Math. **127** (1997), 51-65.

[20] M. Ru, *A defect relation for holomorphic curves intersecting hypersurfaces*, Amer. J. Math. **126** (2004), 215-226.

[21] M. Ru, *Holomorphic curves into algebraic varieties*, Ann. of Math. **169** (2009), 255-267.

[22] B. Shiffman, *On holomorphic curves and meromorphic maps in projective space*, Indiana Univ. Math. J. **28** (1979), 627-641.

[23] N. T. Son, and T.V. Tan, N. V. Thin, *Schmidt's subspace theorem for moving hypersurface targets,* Journal of Number Theory, **186**(2018), 346-369.

[24] P. Vojta, *On Cartan's theorem and Cartan's conjecture*, Amer. J. Math. **119** (1997), 1-17.

[25] P. Vojta, *Diophantine approximation and Nevanlinna theory*, in : J.-L. Colliot-Thélène, P. Swinnerton-Dyer and P. Vojta (eds.), Arithmetic geometry, 111-124. LNM **2009**, Springer-Verlag, Berlin, 2010.

[26] Q. Yan and G. Vu, *Cartan's Conjecture for Moving Hypersurfaces,* Math. Z. (2018). https://doi.org/10.1007/s00209-018-2138-6.

[27] O. Zariski, *Generalized weight properties of the resultant of* $n+1$ *polynomials in* n *indeterminates*, Trans. AMS **41** (1937), 249-265.

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