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A new proof of a theorem of Grauert and Remmert by L_2-methods

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Dedicated to H. Grauert to the occasion of his sixtieth birthday

In this paper we show that on every analytically branched covering $\pi: X \to G$ over a pseudoconvex domain G in the n-dimensional complex space $\mathbb{C}^n$ (cf. Definition 1.1) there exists a holomorphic function f which 'seperates' the sheets. That means that there exists a point $P \in G$ outside the critical locus such that the values of f in the finitely many points which lie over P are pairwise different. The proof, the idea of which was communicated by Siu in the year 1981 to Prof. Grauert and Remmert, depends heavily on special L_2-methods like those which were used by Hörmander in [9] and [10].

Together with a normalisation theorem of Oka this yields that every complex α-space (cf. Definition 1.11) is a normal complex space. In the modern terminology this means that every analytically branched covering is an analytic covering with normal covering space. The theorem was proved in 1958 by Grauert and Remmert in [6] with completely different methods.

1. Basic notions, main theorem and structure of its proof

1. First we recall some basic notations: Let $f: X \to Y$ be a map between topological spaces X and Y. Then f is called *closed* if the image under f of every closed set in X is closed in Y. The map f is called *open* if the image under f of every open set in X is open in Y. The map f is called *finite* if it is continuous and closed, and if every fiber $f^{-1}(y)$, $y \in Y$ consists of finitely many points, only.

Now assume that X and Y are locally compact. Then a continuous map $f: X \to Y$ is called *proper* if every compact set K in Y has a compact inverse $f^{-1}(K)$ in X. It can easily be shown that f is proper iff it is closed and all fibers are compact in X. Hence for locally compact spaces X and Y a map $f: X \to Y$ is finite iff it is proper and all fibers are discrete.

In order to define the concept of analytically branched coverings, we need another topological notion:

Let X be a topological space and A be a closed and nowhere dense subset of X. Then A does not *separate locally* the set X if for every point $P \in A$ and for every connected open neighborhood U of P in X there exists a subneighborhood V of P in U such that $V \setminus A$ is connected.

Now we can give the following *Fundamental definition*:

Definition 1.1. Let X be a locally compact space, G an open domain in $\mathbb{C}^n$ and $\pi: X \to G$ a finite and surjective mapping.

Assume further that there exists an analytic subset A of G with:

(1) $\pi^{-1}(A)$ does not separate locally the set X.
(2) $\pi: (X \setminus \pi^{-1}(A)) \to (G \setminus A)$ is locally topological.

Then $\pi: X \to G$ is called an *analytically branched covering* with *critical locus* A.

Remark. With some easy arguments in elementary topology it can be proved that Definition 1.1 is the same as Definition 3 of [6] for open domains in $\mathbb{C}^n$.

In order to give the reader a better understanding how such an analytically branched covering $\pi: X \to G$ with critical set A looks like, we list some elementary properties:

a) Every point $P \in X$ possesses a countable basis of neighborhoods $U_\nu, \nu \in IN$ such that every $(U_\nu, \pi, \pi(U_\nu))$ is an analytically branched covering. The proof is elementary (cf. [6, p. 259f])

From this we immediately get:

b) π is an open mapping.

c) X has a countable basis of topology. (We can take the countable many components of the π-invers-images of a countable basis of topology of G).

(d) There exists a positive natural number b such that

$$\#(\pi^{-1}(P)) \leqq b \quad \text{for} \quad P \in G \quad \text{with equality if} \quad P \notin A.$$

Hence $\pi: X \to G$ is said to have b *sheets*. The proof again is elementary. At this point it is convenient to give the following definition:

Definition 1.2. Let $\pi: X \to G$ be like in Definition 1.1. A point $P \in X$ is called of *order* k if it has a basis of neighborhoods such that every such neighborhood is an analytically branched covering with k sheets (cf. *a*) and *d*) above). We denote this number k by $o(P)$. The point P is called a *schlicht point* if $o(P) = 1$, otherwise it is called a *branching point*.

e) Now together with d) we get the following structure properties:

For every point $P \in G$ there exists a neighborhood U of P in G such that the following is true:

If $Q_1, \ldots, Q_r$ are the invers images of P the connected components of $\pi^{-1}(U)$ are connected analytically branched coverings (V_i, π, U) with $Q_i \in V_i$, $i = 1, \ldots, r$. Every covering $\pi: V_i \to U$ has $o(Q_i)$ sheets, hence Q_i is the only point in V_i lying over P. Especially we have the formula $\sum_{i=1}^{r} o(Q_i) = b$. Further $\pi: V_i \to U$ is topological iff Q_i is a schlicht point. Hence if all $Q_1, \ldots, Q_r$ are schlicht points, what, for example, is the case if $P \notin A$, the invers image $\pi^{-1}(U)$ of U consists of b connected components which each are mapped topologically to U by π

f) At last let's say something about the critical locus A: It is not uniquely determined, e.g., we can take any nowhere dense analytic subset B of G and get a new critical locus $A' := A \cup B$. It can easily be shown that we can assume A to be empty or pure one (complex) codimensional, since for every point $P \in A$ with $\operatorname{codim}_P(A) \geqq 2$ there exists a neighborhood U of P such that $U \setminus A$ is simply connected, so there cannot lie any branching points over any point of an at least two codimensional irreducible component of A. Hence we can leave away such irreducible components. It even can be shown (cf. [6, p. 265]) that the projection by π of the set of all branching points is an empty or pure one codimensional analytic set in G, which then, of course, is the minimal critical locus.

Next we want to define an analytic structure on analytically branched coverings. The holomorphic functions are defined as the continuous functions on the covering space which are holomorphic in the schlicht points in the sense of domains over $\mathbb{C}^n$, more precisely:

Definition 1.3. Let $\pi: X \to G$ and $\pi': X' \to G'$ be analytically branched coverings and $\Omega \subset X$ an open subset

(1) A continuous function $f: \Omega \to \mathbb{C}$ is called *holomorphic* if for every schlicht point $P \in \Omega$ there exists an open neighborhood $U(P) \subset \Omega$ such that $\pi: U(P) \to \pi(U(P))$ is topological and the function $f \circ (\pi|_U)^{-1}$ is holomorphic in $\pi(U(P))$ The set of such functions is denoted by $\mathcal{O}'(\Omega)$.

(2) A subset $M \subset \Omega$ is called an *analytic set* in Ω if for every point $P \in \Omega$ there exists a neighborhood $U(P) \subset \Omega$ and $f_1, \dots, f_m \in \mathcal{O}'(U(P))$ such that $M \cap U(P) = \bigcap_{i=1}^{m} \{f_i = 0\}$.

(3) A continuous mapping $\psi: \Omega \to X'$ is called a *holomorphic map* if for every $f \in \mathcal{O}'(\Omega')$, where Ω' is an open subset of X', we have $f \circ \psi \in \mathcal{O}'(\psi^{-1}(\Omega'))$. If $\psi: X \to X'$ is bijective and both ψ and ψ^{-1} are holomorphic, it is called *biholomorphic*.

Now we can state the *first main theorem* of this paper

Theorem 1.4. *Let $\pi: X \to G$ be an analytically branched covering with critical locus A. Further assume that G is bounded and pseudoconvex. Let $z_0 \in (G \setminus A)$ and $\pi^{-1}(z_0) = \{x_1, \dots, x_b\}$.*

Then there exists a holomorphic function $f \in \mathcal{O}'(X)$ with pairwise different $f(x_i)$, $i = 1, \dots, b$

2. In this subsection we sketch the structure of the proof of Theorem 1.4. Moreover we reduce the proof of the Theorem 1.4 to the proofs of Theorem 1.7, Lemma 1.8, Proposition 1.9 and Lemma 1.10, which will be given in the Sect. 2 and 3

Without loss of generality we may assume $z_0 = 0$. It suffices to construct an $f \in \mathcal{O}'(X)$ with $f(x_i) = \delta_{i1}$, $i = 1, \dots, b$ (where $\delta_{ij} = 1$ iff $i = j$ and $= 0$ otherwise), since then we can replace x_1 by x_j to get holomorphic functions $f_j \in \mathcal{O}'(X)$ with $f_j(x_i) = \delta_{ij}$, and, at last, we can take an appropriate linear combination of those functions to get the desired f. We further may assume that A is either empty or pure one codimensional in G (cf. f) in the previous subsection or [6, p. 262]).

We introduce some notations: Let $S(A)$ be the singular locus of A and

$$X' := X \setminus \pi^{-1}(S(A)), \quad G' := G \setminus S(A), \quad Y := X \setminus \pi^{-1}(A), \quad H := G \setminus A.$$

Then $\pi: X' \to G'$ and $\pi: Y \to H$ again are analytically branched coverings, the latter has empty critical locus and hence is a domain over $\mathbb{C}^n$.

Now the proof of Theorem 1.4 consists of two parts: In the first part we prove this theorem only for the analytically branched covering $\pi: Y \to H$ instead for $\pi: X \to G$, but with an additional growth condition for the desired holomorphic function (Proposition 1.5). In the second part we use the growth condition to extend this function (after having modified it a bit) to a holomorphic function on X

First part: We want to prove:

Proposition 1.5. *There exists a holomorphic function $h \in \mathcal{O}(Y)$ with*

$$h(x_i) = \delta_{i1}, \quad i = 1, \dots, b \quad \text{and} \quad \int_Y |h|^2 dV < \infty,$$

where dV denotes the Lebesgue measure lifted from H to Y by π

We start with something like the 'C^∞ case' of Proposition 1.5:

Lemma 1.6. *There exists a function $p \in C_0^\infty(Y)$ such that $p(x) \equiv 1$ resp. $p(x) \equiv 0$ in a neighborhood of x_1 resp of $x_2, \dots, x_b$.*

The *proof* of Lemma 1.6 is immediate since $\pi: Y \to H$ is locally biholomorphic. □

We now want to prove Proposition 1.5 by solving a differential equation for the $\bar\partial$ operator with additional growth conditions: More precisely, we want to get a function $u \in C^\infty(Y)$ with the following properties:

$$\bar\partial u = \bar\partial p \quad \text{on } Y \tag{1}$$

$$u(x_i) = 0, \quad i = 1, \dots, b \tag{2}$$

$$\int_Y |u|^2 dV < \infty. \tag{3}$$

The property (2) can be enforced by the modified growth condition

$$\int_Y |u|^2 e^{-\varphi_0 \circ \pi} dV < \infty, \quad \varphi_0(z) := 2n \log|z| : H \to \mathbb{R} \cup \{-\infty\}, \tag{4}$$

what can be seen by pulling the integral, restricted to small neighborhoods of the x_i, $i = 1, \dots, b$, back to H by π. Since H is bounded in $\mathbb{C}^n$ condition (4) also implies condition (3).

If we have once got such an $u \in C^\infty(Y)$ with property (1) and (4) and hence with the properties (1) to (3) we can define $h := p - u$, since then $h \in \mathcal{O}(Y)$, $h(x_i) = \delta_{i1}$ $(i = 1, \dots, b)$ and

$$\int_Y |h|^2 dV = \int_Y |p - u|^2 dV \leqq 2 \int_Y (|p|^2 + |u|^2) dV < \infty$$

So the proof of Proposition 1.5 is reduced to the following (slightly more general) theorem (apply it with $g = \bar\partial p$):

Theorem 1.7. *Let $\pi: Y \to H$ and φ_0 be defined as above. Let $g \in C^\infty(Y)_{(0,1)}$ with $\bar\partial g = 0$ and $\int_Y |g|^2 e^{-\varphi_0 \circ \pi} dV = c_0 < \infty$ Then there exists a function $u \in C^\infty(Y)$ with $\bar\partial u = g$ and a constant $k = k(\text{diam}(H)) \in \mathbb{R}^+$ with*

$$\int_Y |u|^2 e^{-\varphi_0 \circ \pi} dV \leq k \cdot c_0.$$

Here, as usual, $C^\infty(Y)_{(p,q)}$ denotes the set of all (p,q) forms on Y with C^∞ coefficients, and $|\alpha|^2(x) := \sum'_{|I|=p} \sum'_{|J|=q} |\alpha_{IJ}|^2(x)$, where α has the form $\alpha(x) = \sum'_{|I|=p} \sum'_{|J|=q} \alpha_{IJ}(x) dx^I \wedge d\bar z^J$ with resp to the local coordinate system given by π, and $\sum'$ means summation only over strictly increasing multyindices (cf [10, p. 77, 112]) This theorem will be further discussed in the Subsects. 1.3 and 1.4

Second part: We prove that after having multiplied the function $h \in \mathcal{O}'(Y)$ (cf Proposition 1.5) with the square of an appropriate chosen function $t \in \mathcal{O}'(X)$ which vanishes identically on $\pi^{-1}(A)$ and takes the value 1 in x_i, $i = 1, \ldots, b$, we can extend it to a function $f \in \mathcal{O}'(X)$, which then, of course, fulfills $f(x_i) = \delta_{i1}$, $i = 1, \ldots, b$ This assertion, together with the main ideas of its proof, is also contained in [12] as Lemma 1.5 Hence what we do in this second part is not more than to work out those ideas

Since G is a Stein domain, the analytic subset A is the intersection of the zero sets of finitely many holomorphic functions on G (cf. [3, p. 187]), which cannot vanish simultaneously in 0 since $0 \notin A$. If f is such a function we define

$$t(x) := (f(0))^{-1} (f \circ \pi)(x).$$

Now we first prove:

Lemma 1.8. *There exists a holomorphic function $h' \in \mathcal{O}'(X')$ with $h'|_Y = t\,h$.*

In the proof of this lemma, which will be given in Sect. 3, we will use that the analytically branched covering $\pi: X' \to G'$ has the set $A \setminus S(A)$, which only has smooth points, as a critical locus. Over such smooth points of a critical locus, the structure of an analytically branched covering is well known. This will enable us to reduce the proof of Lemma 1.8 to the following:

Proposition 1.9. *Let B be an open subset in $\mathbb{C}^n$ with coordinates $z = (z_1, \ldots, z_n)$ and $B^* := \{z \in B : z_1 \neq 0\}$ Let $f: B^* \to \mathbb{C}$ be holomorphic with $\int_{B^*} |f|^2 d\lambda < \infty$. Then there exists an analytic continuation $\tilde f: B \to \mathbb{C}$ of f to B.*

This result is, in much more general form, obtained in [11, Sect. 1] or in [1, Sect 11], but unfortunately only for two variables (although the proofs given there also work for n variables). For the convenience of the reader we additionally give a short proof of Proposition 1.9 at the end of Sect 3.

—Notice that in this argument the growth condition of the function $h \in \mathcal{O}'(Y)$ is used, because it is needed to yield the growth condition in Proposition 1.9

Now we can finish the second part of the proof of Theorem 1.4 by showing:

Lemma 1.10. *There exists a holomorphic function $f \in \mathcal{O}'(X)$ with $f|_X = t\,h'$.*

The proof of this lemma, which also will be given in Sect. 3, will be done by proving that h' is bounded near points of $X\backslash X'$.

3. The main part of the proof of Theorem 1.4 consists of the proof of Theorem 1.7. It will be given in Sect. 2. Theorem 1.7 is a generalisation to domains over $\mathbb{C}^n$ of the L_2 existence theorems Theorem 4.4.2 of [10] or of Theorem 2.2.3 of [9], each combined with a smoothing argument like in [10, pp. 85–87].

Hörmander also has developed his technics in the case of manifolds (cf. [10, Sect. 5 or 9, Sect. 3]). But for our purpose we cannot use those results. They only yield solutions of the equation $\bar{\partial}u = g$ without additional growth conditions. Further they use that $\varphi \circ \pi \in C^2(Y)$, but our φ has a singularity in the zero point. Because of that reason, we have to approximate our $\varphi \circ \pi$ by $C^2(Y)$ functions $\psi_\nu, \nu \in IN$ and then we can get our u as a weak limes of the solutions u_ν which were obtained with resp. to the ψ_ν. But for the existence of such a weak limes, we need that the u_ν are uniformly bounded in an appropriate way independently of $\nu \in IN$. Hence in our special case those manifold technics wouldn't yield us any section, at all. It does not even seem to be possible to use partial results developed there, since they are not as strong as we need them to yield us such solutions with uniform bounds.

Hence we have to generalize the $\mathbb{C}^n$ case. With two expections, it doesn't seem to be possible to reduce the necessary propositions to those in the $\mathbb{C}^n$ case, but the proofs of those propositions which lead to Theorem 1.7 are nearly verbally the same as those in [10] which lead to the corresponding theorem for domains in $\mathbb{C}^n$. The latter is also stated by Siu and Norguet (cf. Proposition 1.3 of [2]). Nevertheless, since our aim here is not to prove something new but to give a *simple proof* for an already well known theorem, which additionally should be understandable even for those readers who don't have known Hörmander's L_2-methods before, we should give at least the propositions which are the parts of the proof of Theorem 1.7 and the main ideas of their proofs. This should enable the reader to get the ideas of the proofs even without studying [10].

4. Before we start with the discussion of Theorem 1.4, we have to define the concept of a complex α-space (cf. [6, Definition 15]), which is, roughly spoken, a Hausdorff space which locally looks like the covering space of an analytically branched covering:

Definition 1.11. A Hausdorff space R is called *complex α-space* if there exists an open covering $R_\iota, \iota \in I$ with the following properties:

1) For every $\iota \in I$ there exists an analytically branched covering $\pi_\iota : Z_\iota \to B_\iota$ and a topological map $\psi_\iota : R_\iota \to Z_\iota$.
2) If $R_{\iota_1} \cap R_{\iota_2} \neq \emptyset$, the map $\psi_{\iota_2} \circ \psi_{\iota_1}^{-1} : \psi_{\iota_1}(R_{\iota_1} \cap R_{\iota_2}) \to \psi_{\iota_2}(R_{\iota_1} \cap R_{\iota_2})$ is biholomorphic (cf. Definition 1.3).

For every $\iota \in I$, the tripel $(R_\iota, \psi_\iota, \pi_\iota : Z_\iota \to B_\iota)$ is called an *α-chart*. A continuous function $f : \Omega \to \mathbb{C}$, defined on an open subset $\Omega \subset R$, is called *holomorphic* if $f \circ \psi_\iota^{-1} : \psi_\iota(\Omega) \to \mathbb{C}$ is holomorphic for every α-chart $(R_\iota, \psi_\iota, \pi_\iota : Z_\iota \to B_\iota)$ with $\Omega \cap R_\iota \neq \emptyset$.

Now we can state the following theorem of Grauert and Remmert ([6, Satz 32]), which is the *second main theorem* of this paper:

Theorem 1.12. *Every complex α-space is a normal complex space.*

In order to get another version of Theorem 1 12, we recall the notion of an analytic covering (cf [8, p. 133]):

Definition 1.13. A finite surjective map $\pi: X \to Y$ between reduced complex spaces is called an *analytic covering* of Y, if there exists a nowhere dense analytic subset T of Y with the following properties:

a) The set $\pi^{-1}(T)$ is a nowhere dense analytic subset of X.
b) The induced map $\pi: (X \backslash \pi^{-1}(T)) \to (Y \backslash T)$ is locally biholomorphic.

Now Theorem 1.12 is immediately seen to be equivalent with the following:

Theorem 1.14. *Every analytically branched covering is an analytic covering, the covering space of which is a normal complex space.*

For the *proof* of Theorem 1 12, we refer to the second part of the proof of Theorem 31 in [6, p. 297] The main tool of this proof is a theorem corresponding to a local version of Theorem 1.4 (which, in [6], was proved in the Sect. 11–14). The other tools are Oka's normalisation theorem for reduces complex spaces and Riemann's extension theorem for analytically branched coverings [6, p 267]. From these three theorems Theorem 1.12 is obtained easily.

Our proof of Theorem 1.4 is completely different from the proof of the corresponding result in [6], although we have taken some (more basic) facts about analytically branched coverings and their holomorphic functions for it from the first two sections of [6] (those results could probably have been taken also from earlier papers, it is simply more comfortable to take them from [6]).

At last we should tell something about the importance of Theorem 1.12: Historically, there had been two generalisations of complex manifolds: Behnke and Stein introduced spaces equivalent to complex α-spaces, and Cartan introduced spaces equivalent to normal complex spaces With the help of a structure theorem for analytic sets which was proved in [13] in 1953, one can see relatively easy that every normal complex spaces is an α-space. The main purpose of the paper [6] in 1958 was to prove the opposite inclusion, which was done in its main result, which we have stated here as Theorem 1 12. So in those days Theorem 1.12 led together two different concepts of complex spaces.

Even in our days Theorem 1.12 has important applications: For example the proof of the main theorem of [4] from 1983

'*If R is a semiproper holomorphic equivalence relation in a normal complex space then the quotient space X/R is a (weakly normal) complex space*'

depends heavily on Theorem 1 12.

5. The author is very grateful to Prof. Grauert who gave him the main idea of

this proof to work it out. He also wants to thank Prof Diederich for valuable hints for relevant literature.

2. Proof of the L_2-existence theorem

1 The first part of the proof consists of the proof of an inequality, which, in the second part of the proof, will yield us solutions of the $\bar\partial$ equation with the necessary growth conditions by means of the Riesz representation theorem in Hilbert spaces. In order to apply Hilbert space technics, we regard spaces of L_2 forms instead of C^∞ forms, and hence we have to generalize the partial derivatives and the $\bar\partial$ operator in the sense of distribution theory, what will be done in this section, where it is necessary, without mentioning it always. Notice that on analytically branched coverings with empty critical locus the partial derivatives of C^1 functions in direction of the coordinates are well defined since, if we regard such a covering as a manifold with local coordinates given by the projection to the base space, the coordinate transformations are always the identity mapping.

We introduce some notations: Let $\pi: Z \to B$ be an analytically branched covering with empty critical locus and $B \subset \mathbb{C}^n$ with the coordinates $z = (z_1, \ldots, z_n)$ be bounded and pseudoconvex If $\varphi: B \to \mathbb{R} \cup \{-\infty\}$ is semicontinuous from above, we set:

$$L_2(B,\varphi) := \left\{ f: B \to \mathbb{C} : \int_B |f|^2 e^{-\varphi} d\lambda < \infty \right\}$$

$$L_2(Z,\varphi\circ\pi) := \left\{ f: Z \to \mathbb{C} : \int_Z |f|^2 e^{-\varphi\circ\pi} dV < \infty \right\}$$

and define with $L_2(B,\varphi)_{(p,q)}$ resp. $L_2(Z,\varphi\circ\pi)_{(p,q)}$ the set of (p,q)-forms with coefficients in $L_2(B,\varphi)$ resp $L_2(Z,\varphi\circ\pi)$, where dV again denotes integration by the Lebesgue measure λ lifted by π. For elements α of $L_2(B,\varphi)_{(p,q)}$ or of $L_2(Z,\varphi\circ\pi)_{(p,q)}$, we further define $|\alpha|^2$ as in the C^∞-case (cf. Theorem 1.7). Let $\varphi_i: B \to \mathbb{R}$, $i = 1,2,3$ be continuous functions. Then we have

$$L_2(Z,\varphi_1\circ\pi) \xrightarrow{T} L_2(Z,\varphi_2\circ\pi)_{(0,1)} \xrightarrow{S} L_2(Z,\varphi_3\circ\pi)_{(0,2)}$$

where T and S denote the $\bar\partial$ operator (in the sense of distribution theory). Then T and S are densely defined, linear and closed operators between complex Hilbert spaces with $\langle\alpha,\beta\rangle_{\varphi_i\circ\pi} := \sum'_{|J|=i-1} \int_Z \alpha_J \overline{\beta_J} e^{-\varphi_i\circ\pi} dV$, where α and β have the form $\alpha(x) = \sum'_{|J|=i-1} \alpha_J(x) d\bar z^J$ with resp. to the local coordinate system given by π. The proof is straightforward. Let T^* be the adjoint operator of T and D_T, D_S, D_{T^*} the sets of forms for which the corresponding operators are defined.

In order to get the desired inequality, we choose the φ_i, $i = 1,2,3$ as follows: Let $\varphi \in C^\infty(B)$ be strictly plurisubharmonic, i e there exists a positive continuous function $c: B \to \mathbb{R}^+$ with

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar z_k} w_j \bar w_k \geqq c \cdot |w|^2, \quad w \in \mathbb{C}^n \tag{5}$$

Let $\eta_\nu \in C_0^\infty(B)$ with $0 \leqq \eta_\nu \leqq 1$ and $\eta_\nu = 1$ on any compact subset of B if ν is large enough.

Let $\psi \in C^\infty(B)$ with $\sum_{k=1}^{n} |\partial \eta_\nu / \partial \bar{z}_k|^2 \leqq e^\psi$ in B and set

$$\varphi_i := \varphi + (i-3)\,\psi, \quad i = 1, 2, 3$$

Now we can state the desired inequality:

Lemma 2.1. *The following inequality holds for* $f \in C_0^\infty(Z)_{(0,1)}$:

$$\int_Z c \circ \pi - 2|\partial(\psi \circ \pi)|^2)|f|^2 e^{-\varphi \circ \pi} dV \leqq 2\|T^* f\|^2_{\varphi_1 \circ \pi} + \|Sf\|^2_{\varphi_3 \circ \pi} \tag{6}$$

The *proof* of Lemma 2.1 is the same as that of the inequality (4.2.9) in [10, pp. 82–84]. We only have to lift the functions $\varphi, \eta_\nu, \psi, c$ to Z by π. On the contrary the proof simplifies because we have $p = q = 0$ and need to calculate $T^* f$ only for $f \in C_0^\infty(Z)_{(0,1)}$.

The main ideas of that proof are to develop a formula for $T^* f$ yielding a basic inequality which then can be transformed in such a way that inequality (5) can be applied. Since we have $f \in C_0^\infty(Z)_{(0,1)}$ (and not only $f \in D_{T^*} \cap D_S$), we can use integration by parts without getting a non-integral term, and we needn't care about the existence of the derivatives.

Later on in the proof of our existence theorem we will need that, for a function φ with an additional condition, inequality (6) holds for all $f \in D_{T^*} \cap D_S$. So we prove:

Lemma 2.2. $C_0^\infty(Z)_{(0,1)}$ *is dense in* $D_{T^*} \cap D_s$ *for the graph norm*

$$f \to \|f\|_{\varphi_2 \circ \pi} + \|T^* f\|_{\varphi_1 \circ \pi} + \|Sf\|_{\varphi_3 \circ \pi}$$

The *proof* consists of two steps:

First we show that the subset of those forms in $D_{T^*} \cap D_S$ the coefficients of which have compact supports in Z are dense in $D_{T^*} \cap D_S$ for the grph norm. The proof is the same as the first part of the proof of Lemma 4.1.3 in [10, pp. 80–82].

The idea is to show that for $f \in D_{T^*} \cap D_S$, we also have $(\eta_\nu \circ \pi) f \in D_{T^*} \cap D_S$, and that the functions $|S((\eta_\nu \circ \pi) f) - (\eta_\nu \circ \pi) Sf|^2 e^{-\varphi_3 \circ \pi}$ and $|T^*((\eta_\nu \circ \pi) f) - (\eta_\nu \circ \pi) T^* f|^2 e^{\varphi_1 \circ \pi}$ are dominated by $|f|^2 e^{-\varphi_2 \circ \pi}$. That is an easy computation with distributions for the operator S. In order to prove that $(\eta_\nu \circ \pi) f \in D_{T^*}$, we first get by some computation that the inequality $\langle (\eta_\nu \circ \pi) f, Tu \rangle_{\varphi_2 \circ \pi} \leqq c_\nu \|u\|_{\varphi_1 \circ \pi}$ holds for $u \in D_T$ with some positive real constant c_ν. Hence the antilinear operator $D_T \to \mathbb{C}$; $u \to \langle (\eta_\nu \circ \pi) f, Tu \rangle_{\varphi_2 \circ \pi}$ is bounded, and we get with the Hahn Banach Theorem and the Riesz Representation Theorem that there exists an element $v_\nu \in L_2(Z, \varphi_1 \circ \pi)$ with $\langle v_\nu, u \rangle_{\varphi_1 \circ \pi} = \langle (\eta_\nu \circ \pi) f, Tu \rangle_{\varphi_2 \circ \pi}$ for all $u \in D_T$. Hence we have $(\eta_\nu \circ \pi) f \in D_{T^*}$. Now the domination property for the T^* operator can easily be shown by using that of T.

At last it is shown by dominated convergence that $(\eta_\nu \circ \pi) f \to f$ in the graph norm. We remark that dominated convergence is achieved by the special choice of the φ_i, $i = 1, 2, 3$.

Second we show that an element $f \in D_{T^*} \cap D_S$ which has compact support can be approximated by elements of $C_0^\infty(Z)_{(0,1)}$. This assertion can be reduced to the corresponding assertion for domains in $\mathbb{C}^n$: Let $U_\iota, \iota\in I$ be a locally finite covering of Z such that for every U_ι the projection $\pi: U_\iota \to \pi(U_\iota)$ is biholomorphic, and let $t_\iota, \iota\in I$ be a partition of the unity subordinate to the covering $U_\iota, \iota\in I$. Let $f\in D_{T^*}\cap D_S$ have compact support. Then by the linearity of T^*, S and the triangle inequation it suffices to prove that the finitely many $t_\iota f$ which don't vanish identically can be approximated by $C_0^\infty(Z)_{(0,1)}$ forms in the graph norm. So we may assume that f has compact support $K\subset\subset U\subset Z$ and $\pi: U\to\pi(U)=:V$ is biholomorphic. If we now introduce the following Hilbert spaces and operators

$$L_2(V,\varphi_1|_V)\xrightarrow{T':=\bar\partial} L_2(V,\varphi_2|_V)_{(0,1)}\xrightarrow{S':=\bar\partial} L_2(V,\varphi_3|_V)_{(0,2)}$$

with norms $\|\cdot\|_{\varphi_i}$, we have for an α with compact support in U:

$$\|\alpha\|_{\varphi_i\circ\pi}=\|\alpha\circ(\pi|_U)^{-1}\|_{\varphi_i},\quad i=1,2,3$$
$$T'(\alpha\circ(\pi|_U)^{-1})=(T\alpha)\circ(\pi|_U)^{-1},\quad \alpha\in D_T$$
$$S'(\alpha\circ(\pi|_U)^{-1})=(S\alpha)\circ(\pi|_U)^{-1},\quad \alpha\in D_S$$

and hence

$$(T')^*(\alpha\circ(\pi|_U)^{-1})=(T^*\alpha)\circ(\pi|_U)^{-1},\quad \alpha\in D_{T^*}$$

Hence we have $f\circ(\pi|_U)^{-1}\in D_{(T')^*}\cap D_S$ and f can be approximated by $C_0^\infty(U)_{(0,1)}$ forms in the graph norm in Z if $f\circ(\pi|_U)^{-1}$ can be approximated by $C_0^\infty(V)_{(0,1)}$ forms in the graph norm in V. But the latter is just proved in the second part of the proof of Lemma 4.1.3 in [10] (notice that the special forms of the weight functions, which in [10] is different from that in our case, is unimportant for this special proof since everything happens in a compact subset of V).

The main idea of this proof is to regularize an L_2 function f with compact support by convolution with a C^∞ function χ_ε the support of which is contained in a ball of radius ε. Then $f*\chi_\varepsilon\in C_0^\infty(V)$ for sufficient small ε and $f*\chi_\varepsilon\to f$ for $\varepsilon\to 0$ in the L_2 norm. By using some equalities which show how the operators S' and $(T')^*$ act on $f*\chi_\varepsilon$, $f\in D_{(T')^*}\cap D_S$, where the convolution is defined componentwise, we get the same convergence property in the graph norm in V.

2. In this part of the proof of Theorem 1.7 we prove two lemmas, which both are existence theorems of the same kind as Theorem 1.7, but the assumptions on the weight functions are stronger. Further the solutions of the $\bar\partial$ equation they yield are only proved to be L_2 functions. Nevertheless, the following lemma is the most important part of the proof of Theorem 1.7.

Lemma 2.3. *Let $\pi: Z\to B$ and φ be like in Lemma 2.1 and assume that the function c (cf. inequality (5)) is bounded and bounded away from zero. Further let $g\in L_2(Z,\varphi\circ\pi)_{(0,1)}$ with $\bar\partial g=0$.*

Then there exists a function $u\in L_2(Z,\varphi\circ\pi)$ with $\bar\partial u=g$ and

$$\int_Z |u|^2 e^{-\varphi\circ\pi}dV\le 2\int_Z \frac{1}{c\circ\pi}|g|^2 e^{-\varphi\circ\pi}dV<\infty.$$

The *proof* is the same as that of Lemma 4.4.1 of [10].

The main ideas are the following: We take a strictly plurisubharmonic function $s \in C^\infty(B)$ such that $B_a := \{z \in B : s(z) < a\} \subset\subset B$

For fixed $a \in \mathbb{R}$, we can choose $\tilde{\varphi}, \eta_v$ and $\psi \geqq 0$ like in Lemma 2.1 with some additional properties which guarantee that $\tilde{\varphi}_i \equiv \varphi$ in B_a and that in inequality (6) the second term in the first integral vanishes. Since $\psi \geqq 0$ and c is bounded we get from Lemma 2.2 that the inequality (6) of Lemma 2.1 holds for all $f \in D_{T^*} \cap D_S$.

By using that g lies in the kernel of S we conclude from this inequality that $|\langle g, f \rangle_{\tilde{\varphi}_2 \circ \pi}| \leqq k \| T^* f \|_{\tilde{\varphi}_1 \circ \pi}$ for $f \in D_{T^*}$ with $k = 2 \int\limits_Z \frac{1}{c \circ \pi} |g|^2 e^{-\varphi \circ \pi} dV$. Hence $T^*(D_{T^*}) \to \mathbb{C}$; $T^* f \to \langle g, f \rangle_{\tilde{\varphi}_2 \circ \pi}$ is a bounded antilinear operator. So the Hahn Banach Theorem and the Riesz Representation theorem yield an $u_a \in L_2(Z, \tilde{\varphi}_1 \circ \pi)$ with $T^{**} u_a = \bar{\partial} u_a = g$ and $\int\limits_Z |u_a|^2 e^{-\tilde{\varphi}_1 \circ \pi} dV \leqq k$

Since for any a we have $\tilde{\varphi}_i \equiv \varphi$ in B_a we have $\int\limits_{\pi^{-1}(B_a)} |u_a|^2 e^{-\varphi \circ \pi} dV \leqq k$. Hence we can take a sequence $a_j \to \infty$ such that u_{a_j}, $j \in IN$ is weakly convergent on every $\pi^{-1}(H_a)$. Now the limit u has all desired properties.

It is easy to relax the assumptions on φ:

Lemma 2.4. *Let $\pi: Z \to B$ and g be like in the previous lemma. Assume that $\varphi \in C^\infty$ is plurisubharmonic. Then there exists a function $u \in L_2(Z, \varphi \circ \pi)$ with $\bar{\partial} u = g$ and* $\int\limits_Z |u|^2 e^{-\varphi \circ \pi} (1 + |\pi|^2)^{-2} dV \leqq \int\limits_Z |g|^2 e^{-\varphi \circ \pi} dV$.

The *proof* is the same as the first part of the proof of Theorem 4.4.2 of [10]

The idea is only to replace φ by $\varphi + 2 \log(1 + |z|^2)$ and then to apply Lemma 2.3.

3. In this subsection we prove Theorem 1.7, the notations of which we use now.

First we assert that H is pseudoconvex: Indeed, since G was a bounded Stein domain and A was empty or pure one codimensional, H is also a Stein domain by [7, p. 133]. Next we can prove:

Lemma 2.5. *There exists a function $u \in L_2(Y, \varphi_0 \circ \pi)$ such that*

$$\int\limits_Y |u|^2 e^{-\varphi_0 \circ \pi} dV \leqq k \ c_0$$

The *proof* is the same as the second part of the proof of Theorem 4.4.2 of [10].

First we take a plurisubharmonic function s on H such that $H_a := \{z \in H: s(z) < a\} \subset\subset H$. This is possible since H is pseudoconvex. Let $a_j \to \infty$. Then by Theorem 2.6.3 of [10] we can find C^∞ plurisubharmonic functions φ_{a_j} on H_{a_j} such that $\varphi_{a_j} \downarrow \varphi_o$. With Lemma 2.4, applied to $Z = \pi^{-1}(H_{a_j})$, we get a function $u_{a_j} \in L_2(\pi^{-1}(H_{a_j}), \varphi_{a_j} \circ \pi)$ with $\int\limits_{\pi^{-1}(H_{a_j})} |u_{a_j}|^2 e^{-\varphi_{a_j} \circ \pi} (1 + |\pi|^2)^{-2} dV \leqq \int\limits_{\pi^{-1}(H_{a_j})} |g|^2 e^{-\varphi_{a_j} \circ \pi} dV \leqq \int\limits_Y |g|^2 e^{-\varphi_o \circ \pi} dV$. Hence there exists a subsequence of $a_j, j \in IN$ such that the corresponding u_{a_j} converge weakly on every $\pi^{-1}(H_{a_j})$ to a function $u \in L_2(Y, \varphi_o \circ \pi)$ with the desired properties. Notice that here the explicit form of the upper bound in the integral equation in Lemma 2.4 and Lemma 2.5 is needed to guarantee that the upper bound in the above integral inequality is independent of a_j, what is needed to yield $u \in L_2(Y, \varphi_o \circ \pi)$.

Now we are ready to prove Theorem 1.7. All what we still have to show is that the function $u \in L_2(Y, \varphi \circ \pi)$ of Lemma 2.5 actually is C^∞ smooth. Since this is a local problem, we only have to show that if on a pseudoconvex domain Ω in $\mathbb{C}^n$ we have $f \in C^\infty(\Omega)_{(0,1)}$ and a locally square integrable function u on Ω such that $\bar{\partial} u = f$, then $u \in C^\infty(\Omega)$. This is an immediate application of Theorem 4.2.5 and Corollary 4.2.6 of [10].

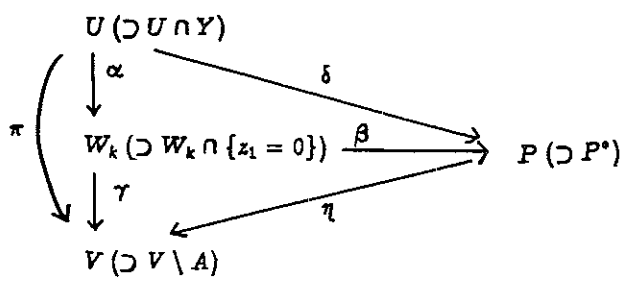
The main idea of the proof is to show by induction that since $\bar{\partial} u = f$ with $f \in C^\infty(\Omega)_{(0,1)}$, the function u lies in Sobolew spaces of arbitrary high order. Then we can use the Sobolew lemma.

3 Proof of the extension theorem

In this section, we use the notations introduced in Theorem 1.4 and in Subsect. 1.2.

1. We prove Lemma 1.8. Let $x \in X' \setminus Y$. It suffices to show that there exists a neighborhood $U(x) \subset X'$ such that the function $t \cdot h$ can be extended continuously into $U(x)$: If x is a schlicht point we can apply the First Riemann Extension Theorem for manifolds, otherwise the continuity is enough.

Since $A \setminus S(A)$ only has regular points and is one codimensional in G', we may assume that there exists a polydisc V with radius $\varepsilon > 0$ around $\pi(x)$ in G' such that $\pi: U \to V$ again is an analytically branched covering with critical locus $A \cap V = \{z_1 = 0\}$. By a dilation we may assume that $\varepsilon = 1$. But the structure of such an analytically branched covering is well known: With [6, pp. 265, 259] there exists a topological map $\alpha: U \to W_k$, where (W_k, γ, V) is given by $W_k := \{(w, z_1, \ldots, z_n) \in \mathbb{C} \times V : w^k = z_1\}$ and $\gamma: W_k \to V$ is the natural projection, such that $\pi = \gamma \circ \alpha$. Further let β be the topological mapping $W_k \to P$; $(w, z_1, \ldots, z_n) \to (w, z_2, \ldots, z_n)$, where P is an n-dimensional unit polydisc with coordinates $\xi = (\xi_1, \ldots, \xi_n)$. Denote further $P^* = P \setminus \{\xi_1 = 0\}$, $\delta := \beta \circ \alpha: U \to P$, $\eta := \gamma \circ \beta^{-1}: P \to V$. Then δ is topological and maps $U \cap Y$ onto P^*, and the mapping η has the form $(\xi_1, \ldots, \xi_n) \to (\xi_1^k, \xi_2, \ldots, \xi_n)$ and hence is holomorphic. The following sketch illustrates our construction:



Now we have:

Lemma 3.1. *1) $h\circ\delta^{-1}$ is holomorphic in P^**
2) $t\circ\delta^{-1}(\xi)=\xi_1^k\cdot t_0(\xi)$, where t_0 is holomorphic in P.
3) $\int_{P^*}|h\circ\delta^{-1}|^2|k\cdot\xi_1^{k-1}|^2 d\lambda(\xi)<\infty$.

Proof. The first assertion follows since locally in P^* we have $h\circ\delta^{-1}=(h\circ\pi^{-1})\circ\eta$. To prove the second assertion we first recall that we had chosen the function t in the form $t=(f(0))^{-1}f\circ\pi$. So we have $t\circ\delta^{-1}(\xi)=(f(0))^{-1}f\circ\eta=(f(0))^{-1}f(\xi_1^k,\xi_2,\ldots,\xi_n)$. Now we have $f(z)=z_1\cdot f_0(z)$ with a holomorphic function $f_0\colon V\to\mathbb{C}$ since f vanishes identically on $A=\{z_1=0\}$ and can be expanded on V into a power series. From this our assertion follows.

To prove the third assertion, it suffices to show that $\int_{U_0}|h|^2dV=\int_{\delta(U_0)}|h\circ\delta^{-1}(\xi)|^2|k\cdot\xi_1^{k-1}|^2d\lambda(\xi)$, where $U_0\subset U\cap Y$ is chosen so that $\pi\colon U_0\to\pi(U_0)$ is bijective. Then our assertion follows since we have $\int_Y|h|^2dV<\infty$ by Lemma 1.5. But we have $\int_{U_0}|h|^2dV=\int_{\pi(U_0)}|h\circ\pi^{-1}|^2d\lambda(z)$ and now the assertion follows from the transformation formula, applied for the mapping $\eta\colon\delta(U_0)\to\pi(U_0)$; $(\xi_1,\xi_2,\ldots,\xi_n)\to(\xi_1^k,\xi_2,\ldots,\xi_n)$. □

If we now apply Proposition 1.9, we get that $(h\circ\delta^{-1})\cdot\xi_1^{k-1}$ can be holomorphically extended from P^* to P. Hence $(t\cdot h)\circ\delta^{-1}=((h\circ\delta^{-1})\cdot\xi_1^{k-1})\cdot(\xi_1\cdot t_0(\xi))$ can also be holomorphically extended. Since δ is topological the proof is complete.

2. We prove Lemma 1.10. If $x\in X\backslash X'$ is a schlicht point, the function $t\cdot h'$ can be extended holomorphically into x by the second Riemann extension theorem for manifolds. If $x\in X\backslash X'$ is a branching point, it suffices to show that h' is bounded around x, since then the function $t\cdot h'$ is continuous there. Over every point $z\in H=G\backslash A$, there are lying exactly b points $x_1,\ldots,x_b$. Let $\omega(w,z)$ be the polynomial of degree b in w the coefficients of which are, for every $z\in H$, the elementary symmetric functions in the b values $h'(x_1),\ldots,h'(x_b)$. Since h' is continuous in X' the coefficients of ω are bounded near points of $A\backslash S(A)$, and hence, by the first Riemann extension theorem for manifolds, can be extended to $G\backslash S(A)$. Hence, by the second Riemann extension theorem for manifolds, they can be extended to G. But then h' has to be bounded near points of $X\backslash X'$.

3. We prove Proposition 1.9. Let $P\in B\backslash B^*$, by a linear transformation we may assume $P=0$. Then we have (cf. [2]) a neighborhood $U(0)\subset B$ and a Laurent series expansion

$$f(z)=\sum_{\nu_2,\ldots,\nu_n\geqq 0}a_\nu z^\nu \quad\text{in}\quad U(0)\cap\{z_1\neq 0\}. \tag{7}$$

Let $D\subset\subset U(0)$ be a polydisc with center 0 and, for $\varepsilon\geqq 0$, $D(\varepsilon):=\{z\in D:|z_1|>\varepsilon\}$. Then we have, by the absolute and locally uniform convergence of the Laurent series:

$$\infty>\int_{D(0)}|f|^2d\lambda\geqq\int_{D(\varepsilon)}|f|^2d\lambda=\sum_\nu|a_\nu|^2\int_{D(\varepsilon)}|z^\nu|^2d\lambda+\sum_{\nu\neq\mu}a_\nu\bar{a}_\mu\overbrace{\int_{D(\varepsilon)}z^\nu\bar{z}^\mu d\lambda}^{=0}$$

Hence if $\varepsilon \to 0$ we get $a_\nu = 0$ for $\nu_1 < 0$, i.e. (7) is a power series expansion of f around the zero point which clearly extends f.

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