Limited Feedback Unitary Matrix applied to MIMO

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Abstract—In spatial multiplexing systems, transmission reliability is enhanced by the full channel state information (CSI) used to design optimum linear precoders, but in practice a full CSI is often unrealistic. Indeed, a higher number of transmitters and/or receivers elevates the coefficient numbers of the estimated channel, and the precision on coefficients depends on the rate of the feedback stream. An interesting alternative is to return a limited amount of information to the transmitter; this led us to design a \( d_{\text{min}} \)-based precoder based on the use of i) a finite codebook known by the receiver and the transmitter and ii) feedback of only one quantized real-valued parameter. The selection criteria employed to find the optimal precoding matrix are presented. Then, the performances about BER are compared under different criteria and amounts of feedback and confronted to Alamouti code.

I. INTRODUCTION

Thanks to rich-scattering wireless channels [1], multiple-input multiple-output (MIMO) systems have recently been found to be attractive because of the huge amount of feedback information to be sent back. The amount of information to be returned can be reduced by using channel statistics [9], [10], partial or quantified CSI [11]–[13].

The present study describes a quantified precoding system based on a limited-feedback unitary precoding derived from Grassmannian subspace-packing problem in order to not compromise the eigenstructure of channel matrix [14]. Indeed, the precoder \( \max-d_{\text{min}} \) is based on the eigen-mode representation of the channel and the optimization of the minimum distance [8]. The \( \max-d_{\text{min}} \) precoder scheme and the codebook based on the unitary precoding matrices are both used in the limited-feedback solution proposed here.

The paper is organized as follows. Section II describes the general precoded and decoded MIMO system and focuses on the optimum \( d_{\text{min}} \) precoder. In Section III, the limited feedback \( d_{\text{min}} \)-based precoder is derived by using the limited feedback scheme proposed by Love and coll. Section IV deals with several criteria based on \( d_{\text{min}} \). The results of simulations carried out with an 8-bit feedback are compared with those of the classical full CSI precoder, criteria and the Alamouti code in Section V. Our conclusions are drawn in Section VI.

II. OPTIMUM \( d_{\text{min}} \)-PRECODER

Let us consider a \((n_T,n_R)\) MIMO system with \( n_R \) receive and \( n_T \) transmit antennas, \( b \) independent data streams, a precoder and a decoder matrices \( F \) \((n_T \times b)\) and \( G \) \((b \times n_R)\), respectively, designed on assuming a perfect knowledge of channel at both sides. The basic system model is:

\[
y = GHFs + Gn
\]

where \( H \) is the \( n_R \times n_T \) Rayleigh flat fading channel matrix with \( \mathcal{N}_c(0,1) \) i.i.d. elements, \( s \) is the \( b \times 1 \) transmitted vector symbol, and \( n \) is the \( n_R \times 1 \) additive white Gaussian noise (AWGN) vector. Let us assume that \( b \leq \text{rank}(H) \leq \min(n_T,n_R) \) and

\[
E[ss^*] = \mathbf{I}_b, \quad R_n = E[nn^*] = \sigma_n^2 \mathbf{I}_{n_R}, \quad E[sn^*] = 0.
\]

By using the following decompositions \( F = F_dF_s \) and \( G = G_v \), the input-output relation (1) can be rewritten as:

\[
y = H_vF_ds + n_v
\]

where \( H_v = G_vHF_s \) is the eigen-mode matrix channel, \( n_v = G_vn \) is the additive noise vector on the channel eigen-mode with the covariance matrix \( R_{n_v} = E[n_vn_v^*] = \sigma_v^2 \mathbf{I}_b \), the unitary matrices, \( G_v \) and \( F_v \), are chosen so as to diagonalize the channel and to reduce dimension to \( b \). The matrix, \( F_d \), results from optimization under a specific criterion.

To calculate \( F_d \) we used the \( \max-d_{\text{min}} \) strategy. The minimum Euclidean distance is defined by:

\[
d_{\text{min}}(F_d) = \min_{s_k,s_l \in \mathcal{C}^b, k \neq l} \|H_vF_d(s_k - s_l)\|.
\]

The \( \max-d_{\text{min}} \) precoder is the solution of:

\[
F_d = \arg \max_{F_d} d_{\text{min}}(F_d)
\]

1. \( \mathcal{N}_c(0,1) \) is the zero-mean and unit-variance complex normal distribution, \( \mathcal{C} \) is the symbol constellation, \( \mathbf{I}_n \) is the identity matrix \( n \times n \), \((.)^* \) is the transpose conjugate and \( \| \cdot \|_F \) is the Frobenius norm.
under the power constraint $\|F_d\|^2 = E_T$. The solution of (5) is difficult, and a very exploitable solution was given in [8] for two independent data streams, $b = 2$ and a 4-QAM. In this case, the matrix $H_v = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2})$ corresponds to the two independent subchannels ($\lambda_1 \geq \lambda_2$ are the two principal eigenvalues of $HH^*$). This solution of $F_d$ is calculated by:

for $0 \leq \gamma \leq \gamma_0$,
$$F_d = F_{r_1} = \sqrt{E_T} \begin{bmatrix} \sqrt{\frac{1 + \sqrt{\lambda_1}}{2}} & 0 \\ 0 & \sqrt{-\frac{1 - \sqrt{\lambda_1}}{2}} \end{bmatrix} e^{i\frac{\pi}{4}}$$  \hspace{1cm} (6)

for $\gamma_0 \leq \gamma \leq \pi/4$,
$$F_d = F_{\text{octa}} = \sqrt{E_T} \begin{bmatrix} \cos \psi & 0 \\ 0 & \sin \psi \end{bmatrix} \begin{bmatrix} 1 & e^{i\frac{\pi}{4}} \\ -1 & e^{-i\frac{\pi}{4}} \end{bmatrix}$$  \hspace{1cm} (7)

where $\gamma = \arctan \sqrt{\frac{\lambda_1}{\lambda_2}}$ is an angle linked to the eigenvalues ratio, $\psi$ is related to the eigen-mode power allocation, and $\gamma_0 \approx 12.28^\circ$ is a constant threshold; moreover, $\psi$ depends on $\gamma$ with $\psi = \arctan \frac{\lambda_2}{\lambda_1}$. Eqs. (6) and (7) can be directly computed to design the max-$d_{\min}$ precoder for only one parameter $\gamma$ to evaluate $F_{r_1}$ or $F_{\text{octa}}$.

III. LIMITED FEEDBACK $d_{\min}$-BASED PRECODER

The Grassmannian theory [15] was employed by Love et al. [14] to compute codebooks, $\mathcal{F}$ with unitary matrix $F_v \in \mathcal{F}$. On condition to assume a Rayleigh MIMO channel, this theory permits one to minimize the average distortion by meeting:

$$E_H \left[ \min_{F_v \in \mathcal{F}} (\|HF_v\|^2_F - \|HF_v\|^2_F) \right].$$  \hspace{1cm} (8)

It is worth recalling briefly how these authors used a practical codebook, $\mathcal{F}$, leading to a family of $N$ matrices. It is defined as:

$$F_{\mathcal{DFT}}$$

where $F_{\mathcal{DFT}}$ is an $nT \times b$ matrix with entry $(k,l)$ equal to

$$(1/\sqrt{nT}) e^{-j2\pi kl/nT}$$ and $\Theta$ is a diagonal matrix given by

$$\Theta = \begin{bmatrix} e^{-j2\pi u_1/nT} & 0 & \cdots & 0 \\ 0 & e^{-j2\pi u_2/nT} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-j2\pi u_N/nT} \end{bmatrix}$$  \hspace{1cm} (10)

where $0 \leq u_1, \ldots, u_N \leq N - 1$.

The coefficients $u_1, \ldots, u_N$ must be chosen for maximizing the minimum distance between the reference matrix, $F_{\mathcal{DFT}}$, and the matrices, $\Theta F_{\mathcal{DFT}}$, as follows:

$$u = \arg \max_{Z \in \mathcal{F}} \min_{1 \leq k \leq N-1} d (F_{\mathcal{DFT}}, \Theta^j F_{\mathcal{DFT}}).$$  \hspace{1cm} (11)

where $u = [u_1, u_2, \ldots, u_N]^T$ and the set $Z = \{u \in Z^N \mid 0 \leq u_k \leq N - 1 \}$.

Since $G_v$ is known at the receiver side, as illustrated in Fig. 1, the feedback of the index matrix from $\mathcal{F}$ and that of the coefficient $\gamma$ are key-parameters.

One should note that, to store the codebook $\mathcal{F}$ at the transmitter/receiver, only the coefficients $u_1, u_2, \ldots, u_N$ are needed; this codebook corresponds to $N log_2 N$ bits. The matrix, $F_v$, can be computed by using the binary codeword index $l$ with $\Theta^{l-1} F_{\mathcal{DFT}}$.

IV. $d_{\min}$GLOBAL SYSTEM

The global system under study associates the max-$d_{\min}$ precoder and the limited feedback unitary precoder. Fig. 1 represents the block diagram. The symbol vector $s = [s_1 s_2]^T$ is precoded by meeting the condition of maximization of $d_{\min}$ distance by the matrix $F_d (F_{r_1}$ or $F_{\text{octa}})$; the only coefficient needed for the matrix calculation is $\gamma$. To guarantee full diversity order, the matrices $F_v$ and $G_v$ are respectively used at the transmitter and receiver; $F_v$ is chosen within the codebook $\mathcal{F}$ by returning the index estimated at the receiver side. To be selected, the unitary matrix, $F_v$, needs to meet at best the chosen criterion.

At first, to choose the unitary matrix, $F_v$, we selected four criteria denoted as follows:

1. The distance calculated here is the chordal distance defined by $d(A,B) = \sqrt{2} ||AA^* - BB^*||_F$.  

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1) Name: min $H_v$

$$\bar{F}_v = \arg \min_{\bar{F}_v \in \mathcal{F}} \left\| H_v - \bar{H}_v \right\|,$$

with $\bar{H}_v = G_v \bar{F}_v$. 

(12)

The max-$d_{\min}$ precoder uses $\bar{H}_v$ to optimize $d_{\min}$.

2) Name: max-$d_{\min}(F_v)$

$$\bar{F}_v = \arg \max_{\bar{F}_v \in \mathcal{F}} \left\{ \min_{\varepsilon_k} \left\| H_v F_d \varepsilon_k \right\| \right\},$$

(13)

where $\varepsilon_k = s_m - s_l$ ($m \neq l$) is the set of differences between the possible transmitted symbol vectors. One should note that, as some vectors are collinear, for a 4-QAM the set of 240 elements can be reduced to 14. The matrix, $F_d$, is calculated with (6) and (7), and $\gamma$ is given by $\gamma = \frac{\arctan \left( \frac{H_v (2)}{H_v (1,1)} \right)}{\pi}$.

Computation of this criterion requires to calculate $14 \times N$ distances.

3) Name: max-$d_{\min}(F_v, \gamma)$

$$(\bar{F}_v, \gamma) = \arg \max_{\bar{F}_v \in \mathcal{F}, \gamma \in [0, \pi/4]} \left\{ \min_{\varepsilon_k} \left\| H_v F_d \varepsilon_k \right\| \right\}.$$ 

(14)

This criterion counterbalances the average distortion by $\gamma$ and $\bar{F}_v$, which are both optimized. It is worth underlining that the search over $\gamma$ has no detrimental effect on the duration of computation; by using a standard optimization function, a satisfying solution is found with only few iterations.

4) Name: unitary – max-$d_{\min}$

$$\bar{F}_v = \arg \max_{\bar{F}_v \in \mathcal{F}} \left\{ \min_{\varepsilon_k} \left\| H_v \varepsilon_k \right\| \right\}.$$ 

(15)

This criterion was proposed in [14]; a single unitary matrix $\bar{F}_v$ is required to optimize $d_{\min}$. As a result, the max-$d_{\min}$ precoder, described in Section II is of no use here.

Fig. 2 presents the $d_{\min}$ probability density function (p.d.f.: normalized histograms) calculated with these criteria where $n_T = 2$ and $n_R = 4$ with 30 000 Rayleigh channel matrices $H$. It shows that the weakest $d_{\min}$ is given by the min $H_v$ criterion, and then by the max-$d_{\min}(F_v)$ one. Concerning the max-$d_{\min}(F_v, \gamma)$ and unitary – max-$d_{\min}$ criterion: for $d_{\min}$, the former is better; on the other hand, for high $d_{\min}$, the unitary – max-$d_{\min}$ criterion works better. These considerations led us to design, in a second step, a 5th criterion, denoted hereafter limited feedback switch $d_{\min}$ (SC-$d_{\min}$). To obtain the best $d_{\min}$, SC-$d_{\min}$ switches from max-$d_{\min}(F_v, \gamma)$ to unitary – max-$d_{\min}$, and conversely. This p.d.f. is illustrated in Fig. 2, by the curve with ‘*’. Table I details the probability of $d_{\min}$ for max-$d_{\min}(F_v, \gamma)$, unitary – max-$d_{\min}$ and SC-$d_{\min}$ criteria when $p_{d_{\min}}(x < 1.2)$ and $p_{d_{\min}}(x > 2.5)$.

V. SIMULATION

A. BER Performances

For the simulation of MIMO system performance, we set the number of antennas to $n_T = 2$ and $n_R = 4$, 30 000 channel matrices $H$ and 1 000 symbol vectors $s$ were generated per $H$, the codebook length was equal to $N = 2^4 = 16$, i.e. 4 bits, and the exact value of the angle $\gamma$ was first fed back, but its quantization will be treated later. Fig. 3 compares the performances between full CSI and limited feedback criteria: min $H_v$, max-$d_{\min}(F_v)$, max-$d_{\min}(F_v, \gamma)$, unitary – max-$d_{\min}$ and SC-$d_{\min}$. It shows that the curves are close in low SNR; on the other hand, when the SNR is increasing, the performances of the max-$d_{\min}$ criteria min $H_v$ and max-$d_{\min}(F_v)$ are degraded. At a BER equal to $10^{-5}$, the loss is up to 2 dB for min $H_v$ and up to 1.3 dB for max-$d_{\min}(F_v)$ when compared to max-$d_{\min}$ full CSI. For max-$d_{\min}(F_v, \gamma)$, unitary – max-$d_{\min}$ and SC-$d_{\min}$ the performance degradation is less than 1 dB. SC-$d_{\min}$ is close to full CSI ($<0.3$ dB). So, with only 4 bits of feedback information and an exact value for $\gamma$, the max-$d_{\min}(F_v, \gamma)$, unitary – max-$d_{\min}$ and SC-$d_{\min}$ are always competitive compared to the system with full CSI.

To explain the good performances of criterion SC-$d_{\min}$, Fig. 6 presents the p.d.f. of the condition number C.N.:

- the normalized histogram of C.N. when unitary – max-$d_{\min}$ is chosen

\[ f_{\text{CN/unitary – max-}d_{\min}}, \]
Let us, now, consider the quantization of the real value $\gamma$.

**B. Quantization of $\gamma$**

Fig. 3 presents the performances with perfect real value $\gamma$. To evidence the impact of $\gamma$ quantization on performances, simulations were carried out with a bit number $n_2$ equal to 2, 3 and 4 and $n_1 = 4$ for the codebook index. Fig. 6 shows the criterion $\max-d_{\text{min}}(F_v, \gamma)$ for different $n_2$. According to the results, $\gamma$ is affected by bit quantization, and BER is degraded. But on condition to have only 4 bits, the performances are decreased by about 0.15 dB compared to the exact transmitted value.

**C. Performance comparisons between SC-$d_{\text{min}}$, full CSI $\max-d_{\text{min}}$ and OSTBC**

Finally, to assess the benefits of our totally limited feedback switch precoder with respect to the performances produced by the full CSI $\max-d_{\text{min}}$ precoder, the SC-$d_{\text{min}}$ precoder and the Alamouti code we performed simulations under the following conditions: $n_r = 2$ and $n_t = 4$, 30 000 $H$, 1 500 symbols per $H$ and $n_1 = 4$ bits and $n_2 = 4$ bits. To meet the bit rate, two 4-QAM were used for both the full CSI $\max-d_{\text{min}}$ precoder and the SC-$d_{\text{min}}$ precoder against two 16-QAM for the Alamouti code. Fig. 7 shows undoubtedly that the SC-$d_{\text{min}}$ precoder behaves nearly alike the full CSI with a loss less than 0.4 dB.
for a BER equal to $10^{-5}$. Compared to the Alamouti code that uses no precoder, the gain obtained by adding only 8 bits of feedback for $SC-d_{\min}$ is equal to 3 dB.

Analysis of results drove us to design a fifth criterion, $SC-d_{\min}$, to switch from a joint optimization of the $max-d_{\min}$ precoder and codebook to a unitary precoder and conversely. This new limited feedback precoder provided very good performances close to those of the full CSI $max-d_{\min}$ precoder with a loss of only 0.4 dB in a (2,4) MIMO system. Compared to the Alamouti code (no CSI at the transmitter side), the gain produced by the $SC-d_{\min}$ was equal to 3 dB, with only an 8-bit return to the transmitter.

VI. CONCLUSION

This investigation were focused on the limited feedback information for the $max-d_{\min}$ precoder was studied. Indeed, many precoders need full CSI at the transmitter side to optimize the power allocation. Then, after the description of the $max-d_{\min}$ precoder, we designed a limited feedback $d_{\min}$-based precoder using a practical codebook to guarantee the full diversity order. To optimize the $d_{\min}$ distance, we proposed four criteria for the selection of the best codebook: $\min H_{\lambda}$, $max-d_{\min}(F_{e})$, $max-d_{\min}(F_{e}, \gamma)$ and unitary $-max-d_{\min}$.

REFERENCES


